Birational Geometry of Cyclic Covers

Thesis submitted in accordance with the requirements of the University of Liverpool for the degree of Doctor in Philosophy

by

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September 2020

Abstract

This thesis concerns the Birational geometry of Fano varieties. The first two chapters are an introduction to Birational geometry, and then specifically the theory of Birational rigidity developed by Pukhlikov and others, as well as an exposition of the method of hypertangent divisors. Using these ideas, we prove two separate results, namely the Birational rigidity of a generic singular cyclic cover, and further show that a generic smooth cyclic cover admits a Kähler-Einstein metric. We finish with a chapter linking our work to previous results, explaining how they link to previous results on fibre spaces, as well as providing some possible areas of future research.

Acknowledgements

I would like to begin by thanking my supervisor Professor Aleksandr Pukhlikov for teaching me the theory of Birational rigidity, guiding me through the study of Algebraic Geometry, as well as being a very supportive and influential figure throughout my time studying at Liverpool. In particular, I owe him a debt of gratitude for every single idea contained within this thesis.

I would like to thank my colleagues at Liverpool University for providing such a friendly and fun atmosphere - in no particular order of importance Lucas, Leticia, Felix, Katie, Tania, Oliver, Jason, Stefano, Daniel, Sam, Rhys, and Aaron. I will miss my time in my office at the rear of the fifth floor of the Department of Mathematical Sciences immensely.

I would like to thank the EPSRC who provided financial support for this wonderful opportunity. Many thanks must also go to my examiners Nicola Pagani and Ivan Cheltsov for reading and providing suggested corrections throughout the entirety of the thesis, as well conducting the Viva in such difficult circumstances. I am hugely appreciative of the work undertaken.

Much mention should go to Nina, Mark, Nick and all the other choristers at the Oriel Singers. I could not have asked for a greater introduction into the Liverpool cultural sphere. I also should mention Barry with whom I spent many an evening warbling away. Many thanks are also due to Matthew and Jane, who provided me with lodging, advice, and some excellent food over the years at their home on the Wirral.

I would like to thank my parents Alex and Gail and my sister Maddy for all their love during this period, Fiona, Steve and their wonderful family, Vir, without whose cajoling I would never have undertaken this work in the first place, and Anja, who patiently sat through hours of my rather amateur exposition and brought me so much joy during this period of my life.

“Ich aber lag am Rande des Schiffs,
Und schaute, träumenden Auges,
Hinab in das spiegelklare Wasser,
Und schaute tiefer und tiefer-”

Heinrich Heine, Seegespenst

Contents

Introduction

0.0.1. Rational varieties and the Lüroth Problem — Algebraic geometry, the field in which this thesis is located, is concerned with the study of zeros of polynomial equations, or alternatively the study of varieties. One main aim is to find a classification of all such objects up to some sort of equivalence relation. Unfortunately, classifying varieties up to biregular isomorphism is a hopelessly thankless task, and so we rely on weaker notions instead. One way to answer this question is by using the notion of *birational equivalence*, which leads us to the subfield of *Birational geometry.* The task can be simply summed up as follows: find invariants under birational maps, which typically consist of both continuous and discrete invariants. One such invariant is the so-called *virtual canonical threshold* - this thesis will discuss this as well as the implications that knowledge of this invariant implies, leading to the theory of *Birational rigidity.*

We however begin by considering a different related question, the so-called *unirationality* problem: we let X be a variety, then the problem asks whether there exists a rational dominant map $\psi : \mathbb{P}^n \dashrightarrow X$. The question whether unirationality implies rationality is known as the *Lüroth problem.* In dimensions 1 and 2, this problem has a positive solution. We use following theorems:

Theorem 0.0.2 (Rationality criterion in dimension 1). — *A curve C is rational if and only if its genus $g_C = h^0(C, \Omega^1_C) = 0$, where Ω^1_C is the sheaf of 1-forms.*

Theorem 0.0.3 (Rationality criterion in dimension 2 (Castelnuovo's Criterion)). — *Let S be a smooth projective surface satisfying*

$$H^0(S, \Omega^1_S) = H^0(S, \omega_S^{\otimes 4}) = H^0(S, \omega_S^{\otimes 6}) = \{0\}$$

where $\omega_S = \Lambda^2 \Omega_S$ is the sheaf of 2-forms. Then S is rational.

On the other hand, for a unirational variety we have the following:

Theorem 0.0.4. — *Let X be a unirational variety defined over the complex numbers. Then $H^0(X, \Omega_X^{\otimes k}) = 0$ for $k > 0$.*

One would therefore hope that we could generalise these to higher dimensions. Unfortunately, this is not the case as we will see. One way of disproving the Lüroth problem in this case is by asking whether a variety is *birationally rigid* or not. Roughly speaking, a variety is birationally rigid if it cannot be transformed into a different variety which is "minimal". We will make this notion precise, and will concern ourselves with the history of this idea in the following.

The origins of the theory of Birational rigidity trace themselves to the work of Fano on algebraic threefolds in the papers [27] and [28]. This was an attempt to expand the Castelnuovo rationality criterion for surfaces to the case of higher dimensions (there is an exposition of this story in [77]). Whilst to modern eyes these papers are riddled with errors, and in particular he was only successful in dealing with particular examples rather than a class of varieties, the ideas contained within them were sound, and certainly generated many advances in the field of Birational geometry.

Inspired by this work, Iskovskikh and Manin wrote the famous paper [35] on the non-rationality of the smooth quartic threefold $V_4 \subset \mathbb{P}^4$, implied by the equality of the automorphism group and the group of birational self-maps, that is, $\mathrm{Bir}(V_4) = \mathrm{Aut}(V_4)$. Since Segre had constructed a smooth unirational quartic threefold in the paper [71], this gave a counterexample to the Lüroth problem in dimension three, i.e. that a unirational variety is always rational.

This was one of three successful attempts to disprove the Lüroth problem that appeared in 1970-1971; the other two are the method of the intermediate Jacobian, by Clemens and Griffiths in the paper [14], and the method of studying the torsion of the third cohomology group, by Artin and Mumford in the paper [2].

0.0.5. The Minimal Model Program — Aside from questions of rationality, another main area of research in Birational geometry since the 1980's has been in the Minimal Model Program (MMP). Heuristically speaking, the aim is to start with a given variety, and birationally transform it into one of three "types", with an output consisting of a variety with canonical class either positive, trivial, or negative. From the point of view of rationality questions, it is very easy to show that all varieties with positive canonical class are non-rational. With a little more work we can show the same for varieties with trivial canonical class. The question, however, for varieties whose output has negative trivial class is far more involved.

The output in the third case is what is known as a *Mori fibre space* - we give the definition:

Definition 0.0.6. — A Mori fibre space is a $\mathbb{Q}$-factorial projective variety X with at worst terminal singularities and a surjective morphism $\phi : X \to Z$ with connected fibres, such that

- The anticanonical class $-K_X$ is ϕ-ample;
- The relative Picard number $\rho(X/Z)$ is 1;
- $\dim Z < \dim X$.

Unfortunately, in general the output of MMP, assuming it is a Mori fibre space, is not unique. However, in the first case, we can (hopefully) apply the theory of Birational rigidity to ascertain some of its properties. We will give a brief overview of this program in Chapter 2.

As of 2020, this theory has been realised in many different contexts, though not in full generality, in particular it has not been proved in every dimension greater than 3. However, the author has no reason to doubt its falsehood, and in any case does not invalidate the utility of the study of Mori fibre spaces in higher dimensions. A very good exposition of the ideas of the MMP is found in the book [21].

0.0.7. The history of Birational Rigidity — The idea of Birational rigidity was then first defined rigorously by Pukhlikov in the paper [51], which grew out of an attempt to refine the methods of the aforementioned Iskovskikh-Manin paper (in particular, any smooth quartic threefold is *birationally superrigid*, a stronger notion than rigidity). Several closely related definitions have since made their way into the literature - we use the definition given in Chapter 2. Birationally rigid varieties are special in that they cannot be birationally transformed into any other Mori fibre space. Since projective space has infinitely many such transformations, it is in particular not rational.

The first variety to be shown to be birationally rigid, though not superrigid, was the quartic threefold with a single non-degenerate singular point; this was proved in the paper [50]. Around the same time, it was shown that the generic intersection of a quadric and a cubic embedded in $\mathbb{P}^5$ was also birationally rigid but not superrigid in the paper [37] - these were papers of Pukhlikov. Cheltsov and Grinenko were able to show in the paper [10] that specific intersections of a cubic and a quadric with a double point were birationally rigid, whilst general such intersections failed to be so. This importantly showed that the property of Birational rigidity is not open in moduli.

Generic Fano hypersurfaces of degree $n \geq 5$ were first studied in [60] - this is where the concept of the technique of hypertangent divisors first made its appearance. The results here were generalised to the case of index 1 complete intersections in the papers [53] and [63], and then to the case of a cyclic cover; an exposition of these ideas can be found in the paper [55], and indeed this thesis improves some of the results contained within. Birational rigidity of arbitrary covers of projective space was then proved in the paper [66]. It should also be mentioned that Johnstone proved the Birational rigidity of singular double quadrics and double cubics in the paper

[38] without the need for hypertangent divisors.

In the field of the classification of threefolds, it was shown that a general member of the famous 95 families of quasi-smooth index one hypersurfaces was birationally rigid in the paper [18] - the genericity condition was later confirmed to be superfluous in [12].

In some cases, it has been shown to be possible to change the genericity condition to one of smoothness, originally conjectured to be the case by Pukhlikov; it was finally proven that every smooth Fano hypersurface of arbitrary dimension $n \geq 4$ is birationally superrigid in the paper [20]. The methods of this paper, involving a extension of the classical inversion of adjunction using multiplier ideal sheaves, was extended further to some smooth and mildly singular complete intersections; see [41, Main Theorem 2] and [45, Theorem 1.1] for proofs of these statements.

The main tool for proving most of these results is the so-called $4n^2$-inequality which was first used indirectly in the (again aforementioned) paper [35], but was codified in the paper [61]. This was recently generalised to the case of a complete intersection singularity, giving much stronger bounds on the multiplicities involved in the paper [65]. Some recent examples of papers making use of this result can be seen in [26], [23] and [57].

Further, it is possible to use the methods to study what happens in the case where the index of the variety is greater than 1. We no longer have the notion of Birational rigidity, as we can always find differing Fano fibre spaces induced by projections. However, we can go some way to describing the possible structures of a rationally connected fibre space using methods from this thesis - see [64] and [19] for some discussion in this area.

Finally we mention the theory of the Birational rigidity of Fano fibre spaces. The origins of the study are difficult to trace, though a good starting point is the paper [52] where the K^2-condition was defined and Del Pezzo fibrations of low degree were studied. As mentioned there is also the paper [55] where pencils of cyclic covers were studied. Most of the papers in this section were written by Pukhlikov, but there are also contributions by Cheltsov, Grinenko, Corti, Reid and de Fernex.

0.0.8. Overview of the Thesis — In Chapter 1 we give a brief review of terminology and results used throughout the thesis, culminating in an introduction to the rationality problem from our point of view. In particular, we begin by giving a description of the setting of the varieties we mostly work with, complete intersections, as well as our chosen form of the notion of multiplicity of a point on a variety. We give a brief discussion on the notion of the *index* of a Fano variety, and how it relates to the birational classification of Fano varieties, as well as a brief introduction to the Minimal Model Programme (abbreviated to MMP), and how Birational rigidity applies to the case where we finish with a variety with negative Kodaira dimension, whilst also remarking that we can and should study varieties in dimensions 4 and higher where MMP is not currently known (but is expected) to work. This sets us up with the techniques required to understand Birational rigidity in the context of the thesis.

The second chapter contains a survey on Birational rigidity, going into detail regarding the methods used later on in the thesis. We begin with an overview of Birational (super)rigidity from the point of view of the threshold of canonical adjunction, before describing the method of maximal singularities, and how it contrasts with "untwisting" as a method to prove the Birational rigidity of a variety. Following the paper [65], we then give a proof based on that of Pukhlikov of the generalised $4n^2$-inequality, which is one of the main tools used to prove Birational rigidity of singular varieties, and how it relates to the original $4n^2$-inequality, also proved by Pukhlikov. Following this we give a recap of the proof of the inversion of adjunction, before we talk about the main method used in this thesis, the method of hypertangent divisors. Throughout we give examples where our methods can be easily applied.

In Chapter 3 we move onto one of the two main results of this thesis, a proof of the Birational superrigidity of a general cyclic cover containing a point of high multiplicity. To do this, we exclude possible maximal centres in turn, depending on their codimension. The hardest case is that when the maximal singularity is a singular point - this is where we use the generalised $4n^2$-inequality in full. We then talk about the regularity conditions required for the use of the method of hypertangent divisors, and prove that the space of parametrising polynomials which are regular is open in the total space. This markedly improves the previous situation where we were only able to allow very simple singularities on such a variety. This result is contained in the paper [29], written by the author of this thesis. We finish with a brief discussion of how this result can be further generalised, to the case where the dimension of singular points on the variety is strictly positive.

In Chapter 4, we discuss the alternate point of view, by which we wish to study the Birational geometry of a cyclic cover, not through its embedding in weighted projective space but instead through projection to a lower dimensional "honest" projective space. For the second main original result of this thesis, we show that the *canonical threshold* of a general *smooth* cyclic cover is bounded below by 1, and discuss the implications of this result in the wider context of K-stability in complex geometry, and give an introduction to this topic. We show this again using the method of hypertangent divisors, and prove that the space of regular defining polynomials is open in the total space.

In the final chapter, we give a brief introduction to the Birational rigidity of Mori fibre spaces, and show how the main result of Chapter 3 has implications for the Birational rigidity of a pencil of cyclic covers. We further discuss possible avenues of research to continue possible applications of hypertangent divisors. In particular, we highlight the efforts to apply the methods in this thesis to higher index situations, as well as further examples of index 1 varieties.

1.

Background Information

In this chapter, we will give an introduction to some of the key concepts from birational geometry. We assume that the reader has knowledge of basic algebraic geometry, for example found in the books [33], [72] and [48]. We will also use the books [30], [36], [43] and [44] for a few technical results, though as before the information contained within these is widely available. We focus on complete intersections and more general Fano varieties, especially concentrating on the distinction between singular and non-singular varieties and the problems they may pose - these are the main objects of study in this thesis.

All varieties will be assumed to be integral and defined over the complex numbers unless otherwise stated. It may well be the case that some of these results do indeed hold over a base of positive characteristic, however we require the use of resolution of singularities throughout this thesis, so we keep things as simple as possible by restricting to the complex case.

1.1. Complete Intersections

Definition 1.1.1. — A projective variety $X = X_{d_1,d_2,\dots,d_k} \subset \mathbb{P}^n$ is called a *complete intersection* if the ideal of X is generated by exactly $\operatorname{codim} X = k$ elements

$f_i(z_0, \ldots, z_n)$, $i = 1, \ldots, k$, each of degree d_i. We may similarly define X embedded in weighted projective space and can also subsequently restrict to affine space; in all of these settings we refer to X as a complete intersection.

We will focus on varieties of this type during this thesis. The reason for this is the following: complete intersection varieties are all Gorenstein, and hence have a canonical divisor that is Cartier.

Definition 1.1.2. — Let X be a projective variety. Let $p \in X$ be a (closed) point, and let $\mathcal{O}_{X,p}$ be the local ring of functions regular at p, with residue field κ and maximal ideal $\mathfrak{m}$. Then we say that X is *non-singular at* p if $\dim_\kappa \mathcal{O}_{X,p} = \dim \mathfrak{m}/\mathfrak{m}^2$, whilst we say p is singular if this is not the case. Further we say that X is non-singular if it is non-singular at every point $p \in X$.

It is then natural to pick a choice of multiplicity of a singular point. Since we will only ever work with complete intersections, we will use the following point of view:

Let $p \in X \subset \mathbb{P}^n$ be a point on a complete intersection X, where $\operatorname{codim} X = k$. Since X is a complete intersection, we may locally at a point p write it as the vanishing of k polynomials $f_1, \ldots, f_k$. We say that p has multiplicity type $|\mu| = \{\mu_1, \mu_2, \ldots, \mu_k\}$ if locally the polynomials $f_1, f_2 \ldots, f_k$ can be decomposed into a sum of homogeneous polynomials as

$$\begin{aligned} f_1 &= f_{1,\mu_1} + f_{1,\mu_1+1} + \ldots + f_{1,d_1} \\ f_2 &= f_{2,\mu_2} + f_{2,\mu_2+1} + \ldots + f_{2,d_2} \\ &\vdots \\ f_k &= f_{k,\mu_k} + f_{k,\mu_k+1} + \ldots + f_{k,d_k} \end{aligned}$$

If it is possible to have a small enough open neighbourhood around p such that p is the only singular point within it, we say that p is an *isolated singularity*. The multiplicity at the point p can easily be seen to be $\mu = \prod_{i=1}^k \mu_i$.

Moving on, from a topological point of view, the singular cohomology of a non-singular complete intersection X can be described by the following theorem, [30, Example 19.3.10].

Theorem 1.1.3. — *Let X be an n-dimensional non-singular complete intersection embedded in $\mathbb{P}^m$. Then $H^i(X, \mathbb{Z}) = H^i(\mathbb{P}^m, \mathbb{Z})$ for $i < n$. The singular cohomology groups of $\mathbb{P}^m$ are well-known to be equal to $\mathbb{Z}$ for $0 \leq i \leq m$ for i even, and zero otherwise, and this gives us the cohomology groups of X.*

This gives us the most basic properties of complete intersections that we will need. Note however that in general the varieties under study will have singularities, necessitating care when we refer to related definitions and theorems.

1.2. Divisors and Linear Systems

In this section, we will describe divisors and linear systems on a projective variety. Studying the Birational geometry of a variety can very frequently be reduced to studying the behaviour of such objects.

Definition 1.2.1. — Let X be a projective variety. A *prime divisor* on X is a closed subvariety of codimension one. A *Weil divisor* is an element of the free abelian group $\operatorname{Div} X$ generated by the prime divisors. We write a divisor D as a sum

$$D = \sum n_i Y_i$$

where the Y_i are prime divisors and the n_i are integers, where only finitely many n_i are non-zero. We say a divisor D is *effective* if all the integers n_i are positive.

Definition 1.2.2. — We say two divisors D and D' are *linearly equivalent*, and write $D \sim D'$ if $D - D'$ is a principal divisor. The group $\operatorname{Div} X/\sim$, where $\sim$ is the equivalence relation defined by linear equivalence is the *divisor class group* of X, and is denoted by $\operatorname{Cl} X$.

Definition 1.2.3. — A *Cartier divisor* on a variety X is defined to be a global section of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$ over X. A *principal divisor* is a Cartier divisor defined by a single rational function $f \in \mathbb{C}(X)$, denoted by (f). We say two Cartier divisors are linearly equivalent if their difference in the group $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ is a principal divisor. The *Picard group* is defined to be the group of all Cartier divisors modulo linear equivalence and is denoted by $\operatorname{Pic} X$.

Definition 1.2.4. — The linear equivalence class of Weil divisors (ω_X) where ω_X is a rational n-form on X is called the canonical divisor class. Any member of this divisor class will be denoted by K_X and is called the *canonical divisor* of the variety X. We can expand this definition to a normal potentially singular variety X; the easiest way is to realise ω_X as the double dual of the sheaf Ω_X^n; we ignore this distinction from this point onward in this thesis - see [69][1.5 and 1.6] for further discussion in this direction.

Definition 1.2.5. — We say that a variety X is *factorial*, if all the local rings of X are UFDs. This implies that all the Weil divisors of X are Cartier, since every subvariety can be written locally as the vanishing of a single function. Similarly, we say that a variety X is $\mathbb{Q}$-factorial if every Weil divisor is some non-zero multiple of a Cartier divisor.

From the point of view of Birational rigidity, it is important that we study varieties that are (at least) $\mathbb{Q}$-factorial. It is not due merely to the presence of singularities that is the "main" obstruction of whether a variety is birationally rigid or not, but its factoriality. Clearly every non-singular variety is factorial, however developing criteria to determine whether a singular variety is $\mathbb{Q}$-factorial is more difficult. This was discussed in a paper of Mella, [46], where he showed that a quartic threefold with at worst quadratic singularities is birationally rigid, as long as it remains $\mathbb{Q}$-factorial. However, in the same paper he showed that a general determinantal quartic threefold is rational.

For Fano threefolds more generally, we can work on a case-by-case basis, often resorting to the topology of the variety in question. Some papers which display some of the ideas involved are by Cheltsov, who was able to bound the number of singular points on threefold hypersurfaces, sextic double solids, nodal quartic threefolds and quartic double solids using a mixture of algebraic and topological arguments; see [9],

[11], [8] and [13] for the respective results.

In this thesis however, we concern ourselves with higher dimensional varieties so can use the following famous theorem proved by Grothendieck which bypasses a lot of the work needed in this case; a modern proof is given as [3, Theorem 7].

Theorem 1.2.6. — *Let X be a variety where every local ring $\mathcal{O}_{X,x}$ is a complete intersection ring. Then X is factorial if the following inequality holds:*

$$\operatorname{codim}(\operatorname{Sing} X) \geq 4.$$

The following theorem allows us to relate a canonical divisor of a non-singular variety with that its restriction to a divisor. It has many applications within the field of Birational geometry.

Theorem 1.2.7 (Adjunction Formula). — *Let D be a non-singular divisor on a non-singular projective variety X. Then in terms of their canonical divisors we have:*

$$(K_X + D)|_D = K_D.$$

Proof. We give a brief proof sketch. Let D and X be as given. Let $i : D \hookrightarrow X$ be the inclusion of D in X with ideal sheaf $\mathcal{I}$. Then there is the conormal exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_X \otimes \mathcal{O}_D \longrightarrow \Omega_D \longrightarrow 0$$

where Ω_X, Ω_D denote the cotangent sheaves of X and D respectively. Taking the determinant of this sequence yields the isomorphism

$$\omega_D \cong \omega_X \otimes \mathcal{O}_X(D) \otimes \mathcal{O}_D.$$

In terms of canonical classes, this is simply

$$K_D = (K_X + D)|_D$$

as claimed. □

Example 1.2.8. — We can easily show that the canonical divisor in terms of hyperplane sections is equal to $(-n-1)H$ for projective space $\mathbb{P}^n$. For a non-singular hypersurface $X_d \subset \mathbb{P}^n$ of dimension d we can use the adjunction formula to show the canonical divisor is equal to $(d-n-1)H$ using the adjunction formula. Inductively applying said formula for $X_{d_1,d_2,\ldots,d_l} \subset \mathbb{P}^n$, tells us that the canonical divisor is equal to $(-n-1+\sum_{i=1}^{l} d_k)H$. Similarly, we also have the following theorem, which requires a bit more work, but gives an analogous result in the weighted projective setting.

Theorem 1.2.9. — *[22, Theorem 3.3.4] Let $X = V_{\underline{d}}$ be a quasismooth weighted complete intersection of multidegree $\underline{d} = (d_1, \ldots, d_s)$ embedded in weighted projective space $\mathbb{P}(a_0, a_1, \ldots, a_n)$. Let $d = \sum_{j=0}^{s} d_j$ and $a = \sum_{i=0}^{n} a_i$. Then $K_X = \mathcal{O}_X(d-a)$.*

We will make use of this theorem to calculate the canonical divisor for a cyclic cover later on.

Definition 1.2.10. — Let D, D' be a pair of divisors on a projective factorial variety X. The *complete linear system associated to D*, is defined to be the set of divisors $E \subset X$ such that $D = E + (f)$ for some principal divisor (f) and is denoted $|D|$. It can be shown that the set $|D|$ is in bijection with the group $(H^0(X, \mathcal{O}_X(D))\backslash\{0\})/\mathbb{C}^*$, and hence has the structure of a projective space. We then define an arbitrary linear system Σ to be a projective subspace of a complete linear system. We say that a linear system is *mobile* if the base locus has codimension 2 or greater, and that a divisor D is *mobile* if its associated linear system $|D|$ is mobile. Let Σ be a linear system on a variety X. Then Σ determines a rational map $\phi_\Sigma : X \dashrightarrow \mathbb{P}^k$ in the following way: If $\{f_1, f_2, \ldots, f_k\}$ is a basis of Σ, then ϕ_Σ maps $x \in X$ to $[f_1(x) : f_2(x) : \ldots : f_k(x)]$.

Throughout this thesis, the importance of mobile linear systems, especially applied to the theory of Birational rigidity cannot be understated. We will discuss this all in Chapter 2. We will also need the following theorem, presented without proof. Note that the second part of the theorem immediately follows the first from *Serre Duality.*

Theorem 1.2.11 (Kodaira Vanishing). — *Let X be a projective nonsingular variety of dimension n over the field $\mathbb{C}$. Let K_X be the canonical divisor on X, and let D be an ample divisor on X. Then*

1. $H^i(X, \mathcal{O}_X(K_X + D)) = 0$, $i > 0$
2. $H^i(X, \mathcal{O}_X(-D)) = 0$, $i < n$

We can also define $\mathbb{Q}$-divisors, by taking the tensor product of $\mathbb{Q}$ with the group $\mathrm{Div}(X)$. Since the coefficients of $\mathbb{Q}$-divisors no longer need be integral, we make a few natural definitions.

Definition 1.2.12. — Let $D = \sum a_i D_i$ be a $\mathbb{Q}$-divisor on a variety X. The *round-up* $\lceil D \rceil$ and *integral part* $\lfloor D \rfloor = [D]$ of D are the integral divisors

$$\lceil D \rceil = \sum \lceil a_i \rceil D_i,$$

$$\lfloor D \rfloor = [D] = \sum [a_i] D_i$$

where for $x \in \mathbb{Q}$ we denote by $\lceil x \rceil$ the least integer greater than or equal to x, by $\lfloor x \rfloor = [x]$ the greatest integer less than or equal to x. The *fractional part* $\{D\}$ of D is defined as

$$\{D\} = D - [D].$$

Definition 1.2.13. — Let X be normal. We say a $\mathbb{Q}$-Cartier divisor D is *big* if some multiple mD induces a birational map onto its image under the map $\phi_{|D|}$ from Definition 1.2.10. We say that D is *nef* if $\deg(D \cdot C) \geq 0$ for all curves $C \subset X$, where deg in some sense counts the number of points of intersection of the two cycles. We make this more explicit in Section 1.3.

Definition 1.2.14. — Considering X and D as above, we say D has *simple normal crossings* at a point $z \in D$, shortened to *D is SNC at z*, if there exists a non-empty Zariski open neighbourhood $U \subset X$ of z such that U is a non-singular subset of X and D is defined by local analytic coordinates of the type

$$\prod_{i=1}^{k} z_i = 0$$

for some $k \leq n$. If D is SNC at every point $z \in D$ we simply say that D is SNC. We say a $\mathbb{Q}$-divisor $D' = \sum a_i D_i$ has *simple normal crossing support* if $\sum D_i$ is an SNC divisor. If we allow the case where $n > k$, we say that D has *normal crossings*.

Using these definitions, the following generalisation of the Kodaira vanishing theorem was proved.

Theorem 1.2.15 (Kawamata-Viehweg Vanishing). — *Let X be a nonsingular proper algebraic variety, and let $D = \sum \alpha_i D_i$ be a nef and big $\mathbb{Q}$-divisor. Assume that the support of the fractional part $\{D\}$ has only simple normal crossings. Then*

$$H^i(X, \mathcal{O}_X(K_X + \lceil D \rceil)) = 0 \quad \forall i > 0.$$

This generalises Kodaira vanishing to the case where our divisor D is no longer integral. Though it feels a somewhat artificial statement, in fact this theorem has a lot of mileage in the topic of the Minimal Model Programme, though this thesis will not venture in that direction. We will however make use of this theorem to prove the so-called *connectedness principle*.

1.3. Intersection Theory

In this section we describe intersection theory on projective varieties, that is how to define the intersection of two subvarieties of a given variety X. This allows us to work out how multiplicities of subvarieties behave under blow up of cycles.

We begin by defining numerical Chow groups. To do this we extend the definition of numerical equivalence of divisors to that of so-called k-cycles. This allows us to perform meaningful intersections of subvarieties of a given variety X.

Definition 1.3.1. — Let X be a factorial n-dimensional quasi-projective variety. We define the group of k-cycles to be the free abelian group generated by subvarieties of dimension k, denoted by $Z_k(X)$. We say a k-cycle $Z = \sum n_i Y_i$ is *effective* if all non-zero coefficents n_i are non-negative.

Definition 1.3.2. — Considering X again as above, we now assume that it is nonsingular. Let k_1 and k_2 be integers less than or equal to n, but such that their sum is greater than or equal to n. We define the *intersection* of a k_1-cycle A and a k_2-cycle

B on X in the following way. Suppose that the intersection of the two cycles is proper: that is that $\dim A \cap B = \dim A + \dim B - \dim X$. We have a map

$$(- \cdot -) : Z_{k_1}(X) \times Z_{k_2}(X) \to Z_{k_1+k_2-n}(X)$$

induced by set-theoretic intersection of such cycles. Taking the set of irreducible components of the image $\{C_1, C_2, \ldots, C_l\} = \mathcal{C}$, we define the scheme-theoretic intersection of A and B to be

$$(A \cdot B) = \sum_{C_i \in \mathcal{C}} \mathrm{mult}_{C_i}(A, B) C_i,$$

where $\mathrm{mult}_{C_i}(A, B)$ is the so-called *intersection multiplicity of A and B along C_i*.

Remark 1.3.3. — There are several (equivalent) ways to define intersection multiplicities. One way which works in full generality is to use Serre's formula [25, Theorem 2.7], though this is a hazardous construction. However, since we only work with complete intersections, by [24, Proposition 18.13], we can assume that our variety X is Cohen-Macaulay, and hence by [30, Proposition 8.2], we have that $\mathrm{mult}_{C_i}(A, B)$ is equal to the length of the ring $\mathcal{O}_{C_i, A\cap B}$.

Remark 1.3.4. — The condition that A and B have proper intersection (i.e. intersect *transversally*) does not need to be checked in any application throughout this thesis: in every case where we take intersections, we are only taking the intersection of *general* cycles, so we can always assume that the intersection is "good".

Remark 1.3.5. — If we relax to the case where X is only $\mathbb{Q}$-factorial, suppose that we have Weil divisors D and E such that nD and mE are Cartier for integers n and m. Then we define the intersection of D and E by

$$D \cdot E = \frac{(nE) \cdot (mD)}{nm}$$

and similarly for lower dimensional cycles.

Definition 1.3.6. — Given a 0-cycle $Z = \sum a_i Y_i$, we define the degree map

$$\deg : Z_0(X) \to \mathbb{Z}$$

which sends $Z \mapsto \sum a_i$. From this, we can define the degree of an arbitrary k-cycle Z' to be $\deg(Z' \cdot H^{n-k})$ where H is a hyperplane divisor of X.

Definition 1.3.7. — Let X be an n-dimensional projective variety. We say that two k-cycles Z and Z' are *numerically equivalent* if for any $(n-k)$-cycle W, we have the following equality:

$$\deg(Z \cdot W) = \deg(Z' \cdot W).$$

Definition 1.3.8. — Let X be a non-singular integral quasi-projective variety. We define the numerical Chow groups $A_k(X)$ to be the groups $Z_k(X)/\sim$ where $\sim$ is the numerical equivalence relation. The intersection product in fact imposes a graded ring structure on the direct sum

$$\bigoplus_{k=0}^{n} A_k(X).$$

We call this ring the Chow ring of X and denote it by $A(X)$.

Remark 1.3.9. — Similarly we take the convention that we can also define the group of k-cocycles $A^i(X) = A_{n-i}(X)$ analogously.

Example 1.3.10. — The Chow ring of $\mathbb{P}^n$ is given by

$$A(\mathbb{P}^n) = \mathbb{Z}[H]/(H^{n+1})$$

where $H \in A_{n-1}(\mathbb{P}^n)$ is the equivalence class of a hyperplane. More generally, the class of a variety of codimension k and degree d is dH^k.

Theorem 1.3.11 (Lefschetz hyperplane theorem). — *Let X be a non-singular projective variety of dimension n, and let D be any effective ample divisor on X. Then the restriction map*

$$r_i : H^i(X, \mathbb{Z}) \to H^i(D, \mathbb{Z})$$

is an isomorphism for $i \leq n-2$ and injective when $i = n-1$.

Corollary 1.3.12. — *Let X be a non-singular complete intersection of dimension ≥ 3 embedded in (weighted) projective space. Then $A^i(X) \cong \mathbb{Z}$ for $i < \frac{\dim X}{2}$.*

Proof Sketch. For a k-cocycle Z on a non-singular projective variety X, there is a so-called *class map* sending Z to its image in $H^{2k}(X, \mathbb{Z})$. We say that two k-cocycles are *cohomologically equivalent* if their images under this map are equal. We can then use the Lefschetz Theorem to derive this result; see [30, Example 19.3.10] for the cohomology groups of a complete intersection. Cohomological equivalence implies numerical equivalence, which gives us the result. □

Definition 1.3.13. — We also will define $A^i_{\mathbb{R}}(X) := A^i(X) \otimes \mathbb{R}$, $A^i_+(X)$ the closure of the cone in $A^i_{\mathbb{R}}(X)$ generated by classes of effective cycles (containing pseudo-effective cycles), and the pseudo-effective cone, $A^i_{\mathrm{mob}}(X)$, the closure of the cone generated by classes of mobile divisors.

Definition 1.3.14. — Let $\pi : \tilde{X} \to X$ be the blowing up of a variety X at a subvariety Y, and let $D \subset X$ be a divisor. Let Z be the image of the exceptional locus, and suppose that $D \not\subset Z$. We define the *strict transform* of D to be the closure of the inverse image $\pi^{-1}(D \backslash Z)$. Similarly, we can define the *strict transform of a linear system* Σ to be the closure of the inverse image $\pi^{-1}(\Sigma \backslash Z)$. We can extend this definition to arbitrary k-cycles not contained in Z similarly.

Definition 1.3.15. — Considering now the blow up of a quasi-projective variety X along an irreducible cycle B of codimension ≥ 2 with exceptional divisor $E(B)$, let $Z = \sum a_i Z_i$, $Z_i \subset E(B)$ be any k-cycle where $k \geq \dim B$. We define the *degree* of Z setting

$$\deg Z = \sum_i a_i \deg\left(Z_i \cap \sigma_B^{-1}(b)\right)$$

where $b \in B$ is a generic point on B, $\sigma_B^{-1}(b) \cong \mathbb{P}^{\operatorname{codim} B - 1}$ and the right-hand side degree is equal to the degree defined in Definition 1.3.6.

Now that we have all the main definitions of the intersection theory we use in this thesis, we will highlight the following very important lemma.

Lemma 1.3.16. — *Let D and Q be two different prime Weil divisors on a quasi-projective variety X, again let $\sigma_B : X(B) \to X$ be the blow up of an irreducible cycle B of codimension ≥ 2 with exceptional divisor $E(B)$ and let $D^B, Q^B, (D \cdot Q)^B$ be the strict transforms of the divisors and their intersection on $X(B)$. Then:*

1. *Assume that* $\operatorname{codim} B \geq 3$. *Then*

$$D^B \cdot Q^B = (D \cdot Q)^B + Z$$

where $\operatorname{Supp} Z \subset E(B)$ *and*

$$\operatorname{mult}_B(D \cdot Q) = (\operatorname{mult}_B D)(\operatorname{mult}_B Q) + \deg Z$$

2. *Assume that* $\operatorname{codim} B = 2$. *Then*

$$D^B \cdot Q^B = Z + Z_1$$

where $\operatorname{Supp} Z \subset E(B)$, $\operatorname{Supp} \sigma_B(Z_1)$ *does not contain* B *and*

$$D \cdot Q = \{(\operatorname{mult}_B D)(\operatorname{mult}_B Q) + \deg Z\}B + (\sigma_B)_* Z_1.$$

Proof Sketch. The first part is almost trivial. For the second, we can assume that B is a surface by taking a generic point $b \in B$, letting $S \ni b$ be a germ of a nonsingular surface in general position in B, S^B its proper inverse image on $X(B)$. This reduces the question to the intersection of two irreducible curves at a non-singular point on a surface in terms of its blowup. □

Remark 1.3.17. — We note the distinction between the two cases where we blow up a cycle of codimension 2 and one of higher codimension; this is crucial to the proof of the $4n^2$-inequality in Section 2.3.

Finally, we need the following for the proof of Theorem 2.3.9. This was proved in the paper [74] by Suzuki, and was an extension of the original so-called *cone method*.

Theorem 1.3.18. — *[Pukhlikov's Lemma] Let $X \subset \mathbb{P}^N$ be a non-singular complete intersection of codimension $l \geq 1$, $S \subset X$ a subvariety of codimension $a \geq 1$ and $B \subset X$ a subvariety of dimension al, where $N \geq (l+1)(a+1)$ holds. Then the inequality*

$$\mathrm{mult}_B S \leq m$$

holds, where $m \geq 1$ is defined by the condition $S \sim mH_X^a$ where $H_X \in A^1X$ is the class of a hyperplane section of X.

This is [74, Proposition 2.1], and is a generalisation of the previously used method, known as the so-called *cone method*. The proof of this very similar in spirit to the proof of the original for the hypersurface case, [62, Proposition 3.6].

1.4. Birational Classification of Varieties

At this point, we have the necessary language in order to describe and attack the problem of the classification of varieties up to birational equivalence. We recall first of all the definition of a birational map:

Definition 1.4.1. — Let $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ be two projective varieties. A *correspondence* Z from X to Y is a relation given by a closed algebraic subset $Z \subset X \times Y$. Z is said to be a rational map if Z is irreducible and there is a Zariski open set $X_0 \subset X$ such that every $x \in X_o$ is related by Z to one and only one point of Y. Z is said to be a birational map if $Z \subset X \times Y$ and $Z^{-1} \subset Y \times X$ are both rational maps. If there exists a birational map between varieties X and Y, we say that X and Y are *birational* to one another. Equivalently, we can say that two varieties X and Y are birationally isomorphic to each other if and only if their respective function fields $K(X)$ and $K(Y)$ are as well.

Note that instead of writing out the correspondence $Z \subset X \times Y$ each time, we will abbreviate this to a map $\phi : X \dashrightarrow Y$.

In particular, we can talk about the *rationality* of a variety; we say a variety V is rational if it is birational to projective space $\mathbb{P}^n$ for some n.

The most important theorem we have in relation to the rationality problem is the following famous theorem due to Hironaka (1964).

Theorem 1.4.2. — *Let X be an irreducible variety, and let $D \subset X$ be an effective Weil $\mathbb{Q}$-divisor on X. Then:*

1. *There is a (not-necessarily unique) projective birational morphism*

$$\mu : \tilde{X} \to X$$

composed of blow ups of subvarieties of X contained in $\operatorname{Sing}(X)$ where $\tilde{X}$ is non-singular and μ has divisorial exceptional locus $\operatorname{exc}(\mu)$ such that

$$\mu^{-1} \operatorname{Supp}(D) + \operatorname{exc}(\mu)$$

is a divisor with SNC support.

2. *We can further assume that μ is an isomorphism away from the locus where D does not have simple normal crossing support.*

This is known as a *log resolution* of X. If we take $D = 0$, then we call it a *resolution of singularities* of the variety X. In particular, every complex projective variety X is birational to a non-singular projective variety X'.

Remark 1.4.3. — We should note that the theorem applies more generally than in our case, to more general fields of characteristic 0. However, this is the form of the theorem that is most useful for this thesis.

Another consequence of the theorem on the resolution of singularities is that it allows us to define a notion of singularity based on the exceptional divisors of any resolution of singularities in the following way, by defining the notion of a *pair*:

Definition 1.4.4. — Let X be a normal variety with a Weil $\mathbb{Q}$-divisor $D = \sum d_i D_i$ such that $K_X + D$ is $\mathbb{Q}$-Cartier on X (note that we allow the coefficients d_i to be completely general). We call (X, D) a *pair*.

Remark 1.4.5. — From this we can define a log resolution of the pair (X, D) in exactly the same way as above.

Now let $\mu : Y \to X$ be a birational morphism, where Y is normal. Then there are rational numbers

$$a(E) = a(E, X, D) \in \mathbb{Q}$$

attached to each prime divisor $E \subset Y$ having the property that

$$K_Y \equiv \mu^*(K_X + D) + \sum a(E).E,$$

where the sum runs over every prime divisor of Y. Note that the right hand side is not unique, as we allow non-exceptional divisors in the summation. Therefore we adopt the following:

A non-exceptional divisor E appears in the right hand side if and only if $E \equiv f_*^{-1} D_i$ for some summand D_i in D, with coefficient $a(E, X, D) = -d_i$. The push-forward in this definition is defined as in [25, Definition 1.19] and in this case maps Weil divisors to Weil divisors.

Definition 1.4.6. — $a(E, X, D)$ is called the *discrepancy* of E with respect to the pair (X, D). Note that if $f : Y' \to X$ is another birational morphism, by invariance of the function field of X, if E' is the birational transform of E on Y', then we have the equality $a(E, X, D) = a(E', X, D)$.

Remark 1.4.7. — We also may assume that the divisor in the pair (X, D) may well be empty. In this case, we simply drop the D from the definition of discrepancy and write $a(E, X)$. Secondly, if $f : Y \to X$ is any birational morphism to a pair (X, D), then there exists a unique divisor D_Y on Y such that

$$K_Y + D_Y = f^*(K_X + D) \text{ and}$$
$$f_*(D_Y) = D.$$

Definition 1.4.8. — Let (X, D) be a pair. Then we define

$$\text{discrep} := \inf\{a(E, X, D) | E \text{ is exceptional with non-empty centre on } X\}$$
$$\text{totaldiscrep} := \inf\{a(E, X, D) | E \text{ has non-empty centre on } X\}$$

Example 1.4.9. — Suppose $E \subset X$ is a divisor different from any of the D_i, then $a(E, X, D) = 0$, and so totaldiscrep$(X, D) \leq 0$. Similarly, if E is obtained by blowing up any non-singular codimension 2 subvariety, then by [33, Exercise II.8.5], $a(E, X, D) = 1$, so discrep$(X, D) \leq 1$.

We now have all the language we need to define what we mean by the singularity of a pair.

Definition 1.4.10. — Let X be a normal variety and $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. We say that (X, D) is

terminal	if discrep$(X) > 0$
canonical	if discrep$(X) \geq 0$
log terminal (plt)	if discrep$(X) > -1$
Kawamata log terminal (klt)	if discrep$(X) > -1$ and $\lfloor D \rfloor \leq 0$
log canonical	if discrep$(X) \geq -1$

These are the bread and butter definitions for singularities, and play a large role in the classification according to the *minimal model programme*, which we will discuss in relation to Birational rigidity shortly.

We now list a few more properties that a variety can have in relation to its Birational geometry.

Definition 1.4.11. — Let X be a projective variety, and let $\tilde{X}$ be a resolution of X. Let $R(X)$ be the ring

$$R(X) = \bigoplus_{n \geq 0} H^0(X, nK_{\tilde{X}}).$$

We call this the *canonical ring* of X. We define the Kodaira dimension $\kappa(X)$ of the variety X to be the dimension of the Proj of the ring $R(X)$, if it is greater than or equal to 0, and $-\infty$ otherwise.

Theorem 1.4.12. — *Let X be a smooth projective variety. Then the canonical ring of X, $R(X)$, and hence the Kodaira dimension $\kappa(X)$ is a birational invariant.*

The birational invariance of $R(X)$ follows from the invariance of plurigenera, proved in [33, Chapter 2, Theorem 8.19].

We now come to the definition of the main objects of study in the field of birational geometry.

Definition 1.4.13. — A non-singular projective variety X is called a

- variety of general type if its canonical divisor K_X is ample;
- Calabi-Yau variety if its canonical divisor K_X is numerically trivial;
- Fano variety if its anticanonical divisor $-K_X$ is ample.

Note that alternative definitions are sometimes used, especially in the case of Calabi-Yau varieties, where sometimes we might require the vanishing of all the intermediate cohomology groups $H^i(X, \mathcal{O}_X)$. However, this is the most relevant to our setting.

The case of a variety of general type can be dealt with from a birational perspective by the following theorem:

Theorem 1.4.14. — *Let $\phi : X \dashrightarrow Y$ be a birational map where X and Y are non-singular varieties such that the canonical divisors K_X and K_Y are ample. Then ϕ is an isomorphism.*

Proof. We reproduce the proof from [5] as follows: A birational map between non-singular varieties X and Y induces an isomorphism between their canonical rings, and hence one between the spaces $H^0(X, mK_X)$ and $H^0(Y, mK_Y)$ for every $m \geq 0$. However, since K_X and K_Y are also ample, we have

$$X \cong \operatorname{Proj}\left(\bigoplus_{n\geq 0} H^0(\mathcal{O}_X(nK_X))\right) \cong \operatorname{Proj}\left(\bigoplus_{n\geq 0} H^0(\mathcal{O}_X(nK_X))\right) \cong Y.$$

$\square$

Remark 1.4.15. — In the sense that the degree of a hypersurface of general type is unbounded above, we can state that "most" varieties are of this kind. Given the above theorem, we note that study of varieties of general type are done using biregular methods.

We do not touch on the Birational geometry of Calabi-Yau varieties, though this is a rich topic with a lot of research in this area. We are most interested in Fano varieties, and generalise their definition to include the following:

Definition 1.4.16. — If a normal projective variety X has terminal singularities, and some positive integral multiple $-nK_X$, $n \in \mathbb{N}$ of the anticanonical Weil divisor $-K_X$ is an ample Cartier divisor, then we call X a *singular Fano variety.*

Note that in the case considered above, the Kodaira dimension is equal to $-\infty$. The most important properties of Fano varieties can be summed up in the following theorem.

Proposition 1.4.17. — *[36, Theorem 2.1.2] Let X be an n-dimensional singular Fano variety with klt singularities, and let $f : Y \to X$ be a resolution of singularities. Then:*

1. $H^i(X, \mathcal{O}_X) = H^i(Y, \mathcal{O}_Y) = 0$ *for every* $i > 0$*;*
2. $\mathrm{Pic}(X)$ *and* $\mathrm{Pic}(Y)$ *are finitely generated torsion-free* $\mathbb{Z}$*-modules;*
3. *Numerical and linear equivalence coincide on the set of Cartier divisors on both* X *and* Y*;*
4. $\kappa(Y) = -\infty$.

Proof. We need a lemma, proved in [43, Chapter 4]:

Lemma 1.4.18 (Injectivity lemma). — *Let $f : Y \to X$ be a finite surjective morphism of irreducible projective varieties where X is normal, and let $\mathcal{L}$ be a coherent sheaf on X. Then the natural homomorphism*

$$H^j(X, \mathcal{L}) \to H^j(Y, f^*\mathcal{L})$$

induced by f is injective.

1) follows immediately from Kawamata-Viehweg vanishing. For 2), we consider the exponential sequence of sheaves over $\mathbb{C}$:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y^* \longrightarrow 0$$

Taking the induced long exact sequence of this sequence:

$$\begin{tikzcd} \cdots \arrow[r] & H^1(X, \mathcal{O}_Y) \arrow[r] \arrow[d, equal] & H^1(Y, \mathcal{O}_Y^*) \arrow[r] & H^2(Y, \mathbb{Z}) \arrow[r] & H^2(Y, \mathcal{O}_Y) \arrow[r] \arrow[d, equal] & \cdots , \\ & 0 & & & 0 & \end{tikzcd}$$

using the isomorphism $\operatorname{Pic} Y \cong H^1(Y, \mathcal{O}_Y^*)$, as well the finite generation of $H^2(Y, \mathbb{Z})$ gives us the statement over Y. Using the injectivity lemma above then proves it for X.

To prove 3), one implication is immediately obvious (this is Exercise V.1.7 in [33]). Suppose now that D is a Cartier divisor on Y, and suppose $D \equiv_{\text{num}} 0$, where $\equiv_{\text{num}}$ denotes numerical equivalence. Then by the Hirzebruch-Riemann-Roch theorem we get $h^0(Y, \mathcal{O}_Y(D)) = h^0(Y, \mathcal{O}_Y) = 1$. Therefore, there exists an effective divisor $D_0 \in |D|$, the divisor of zeros of an arbitrary non-zero section of $\mathcal{O}_Y(D))$. We then write

$$D_0 = \sum_{i=1}^{k} n_i D_i$$

where the D_i are prime divisors on X and the coefficients n_i are all positive. Since Y is projective, the result follows by intersecting with $n-1$ general hyperplanes, none of which are contained in $D_2, \ldots, D_k$ and intersect D_1 transversally. We then obtain:

$$0 = (D \cdot H^{n-1}) = (D_0 \cdot H^{n-1}) = \sum_{i=1}^{k} n_i (D_i \cdot H^{n-1}) \geq n_1 D_1 \cdot H_{n-1},$$

which shows that n_1, and hence every n_i, is equal to zero. Therefore, D is linearly equivalent to the zero divisor and we are done. This similarly holds for X again using the injectivity lemma.

To show the Picard group is torsion free, we suppose that there exists some torsion element $D \in \operatorname{Pic} X$ so that $\alpha D \equiv_{\text{num}} 0$ for some nonzero integer α. This implies that $(\alpha D \cdot C) = \alpha(D \cdot C) = 0$ for all curves C on X, so that $(D \cdot C) = 0$ for all curves C and hence that $D \equiv 0$. By the equivalence of numerical and linear equivalence, we get that D is equal to 0 in $\operatorname{Pic}(X)$.

The last part follows by the definition of Kodaira dimension. □

Corollary 1.4.19. — *For a Fano variety X with klt singularities there exists a greatest rational number $r \in \mathbb{Q}_+$ such that $K_X = -rH$ for some ample divisor H (the fundamental divisor). We call this number $r = r(X)$ the* index *of the Fano variety X.*

Proof. This follows from the second part of the theorem above. □

Remark 1.4.20. — In fact, this theorem also tells us that the Picard group $\operatorname{Pic} X$ is in fact isomorphic to the group of $(n-1)$-cycles (under the first Chern class map). This is not true in general; if for example if we take X to be an irreducible plane cubic with a node, then it can be shown that $c_1 : \operatorname{Pic}(X) \to A_{n-1}(X)$ is not injective (See [25, Exercise 1.35] for a proof of this fact.).

Proposition 1.4.21. — *Let X be a n-dimensional Fano variety with klt singularities of index r and let H be a fundamental divisor. Then*

$$H^i(X, \mathcal{O}_X(mH)) = 0 \quad \forall\, i > 0,\, m > -r.$$

Proof. This follows from the Kawamata-Viehweg vanishing theorem. □

Corollary 1.4.22. — *The index of a Fano variety with klt singularities X,* $\operatorname{ind}(X)$*, is not greater than* $\dim X + 1$.

Proof. By the Riemann-Roch theorem, $\chi(\mathcal{O}_X(mH))$ is a polynomial of degree $\dim X$, where χ is the Euler characteristic. By the above proposition, the roots of this polynomial are integers

$$n = -1, -2, \ldots, 1 - \lceil r \rceil,$$

where $\lceil r \rceil$ is the smallest integer greater than r, from which we get $r \leq \dim X + 1$. $\square$

From here, we have the following well-known pair of theorems. In some sense, they hint that high index varieties are nearly always rational.

Theorem 1.4.23. — *[39, Theorem 1.1, Theorem 2.1] Let X be a non-singular Fano variety of dimension n. If the index $\operatorname{ind}(X) = n + 1$, then $X \cong \mathbb{P}^n$. Similarly if $\operatorname{ind}(X) = n$, then X is necessarily a quadric. In both cases, the variety X is rational.*

On the other hand, we can ask about what happens in the opposite case, namely when the index of a Fano variety is low. When we are dealing with index 1 varieties, we can prove its non-rationality by looking at its Birational rigidity.

1.4.24. Rational Connectedness —

Definition 1.4.25. — We say a variety X of positive dimension is *rationally connected* if any two points on X can be joined by a rational curve. That is, for every two points a and b on X, there is a map

$$\phi : C \to X$$

such that $\phi(0) = a$ and $\phi(\infty) = b$ for distinguished points $0, \infty \in C$, where C is a curve whose normalisation is isomorphic to $\mathbb{P}^1$.

Example 1.4.26. — $X = \mathbb{P}^n$ is clearly rationally connected; any two points can be joined by a line l. It is also a theorem that the quadric surface $Q \subset \mathbb{P}^3$ is rationally connected.

In fact we can say much more:

Theorem 1.4.27. — *[42, 78] Fano varieties with at worst klt singularities are rationally connected.*

From our point of view this is practically the first step to proving the Birational (super)rigidity of Fano varieties, as it allows us to define the *threshold of canonical adjunction* and show that it is finite. We will do this in the next chapter.

We can also define rational connectedness in the relative case.

Definition 1.4.28. — A surjective morphism $\pi : X \to S$ of projective varieties is called a *rationally connected fibre space* if the base S and a fibre of general position $\pi^{-1}(s)$, $s \in S$ are rationally connected varieties. By [31, Lemma 3], it then follows that X is also rationally connected.

Studying and classifying rationally connected varieties is another of the key problems of Birational geometry. It is clear that this notion is closely related to that of a Mori fibre space - indeed it can be shown that an Mori fibre space, as defined in the introduction is the end point of MMP applied to a rationally connected variety. We will discuss this notion in the next section.

1.5. The Minimal Model Program, the Sarkisov Program and Birational Rigidity

We detour here to give a (very) brief introduction to MMP, mentioned in the introduction, and its relationship with the theory of our interest, Birational rigidity.

The aim of MMP is to assign to every variety a so-called model birational to the original which is as "nice" as possible. Beginning in dimension one, we can show that two non-singular curves are birational if and only if they are isomorphic. In dimension 2, we can show that any birational map can be factored into a sequence of finitely many blow-ups, followed by blowing down finitely many times [72, Chapter 2, Section 4]. It can also be shown that the exceptional locus of any blow up of a surface is a so-called *minimal*, or (-1)-curve [33, V.3.1], that is a curve with self-intersection number equal to -1. Conversely, we can show that any (-1)-curve can also be blown down. [33, V.5.7]. From this, we can immediately deduce that any surface is birational to one without any minimal curves. This is a simplification of the ideas that led to Castelnuovo's proof for his criterion of rationality for surfaces. The question, then, is whether we can generalise this to higher dimensions.

Initial works of Mori [47] and Reid [68] set out ideas to build the *program* that would be able to enact this idea in dimension 3. The main idea is that it is possible to replace the condition of having no (-1)-curves by asking that the canonical bundle of a minimal variety to be numerically effective (nef) in the case where the variety was of general type, i.e has positive intersection with all curves lying on the variety. Alternatively, in the case where our variety V had Kodaira dimension equal to $-\infty$, we ask that the anticanonical bundle is nef instead. This is the Fano case, and the one where our theory of Birational rigidity applies.

Noting that in general, the output of MMP for the Kodaira dimension equal to $-\infty$ case is not unique, the theory of Birational rigidity aims to discern when the opposite holds. In addition, thus far we have only been able to prove the validity of MMP in general in dimension 3. Due to this, studying the Birational geometry of varieties in higher dimensions requires us to study under a slightly more general setup, as we will see.

Similarly, we can ask whether birational maps can also be factored in a systematic way as well as the Sarkisov program of factoring birational maps. The main idea is that any birational map between to Mori fibre spaces can be factored into one of four kinds of *links*. For threefolds, this has become a very powerful tool of the study of their explicit birational geometry. That every birational map could be factored into finitely many of these links was proved in the paper [15], and many of the ideas within could be applicable directly to the study of the explicit Birational geometry of threefolds. Unfortunately, though we have a similar result in higher dimensions proved in the paper [32], this is by no means a constructive result, and thus we cannot in general use the same methods.

Indeed, one way of proving the Birational rigidity of an Mori fibre space is to ask whether there are any initial links to another variety. Our approach is different. The method of maximal singularities is (relatively) advantageous in the setting of absolute Mori fibre spaces, that is Mori fibre spaces with a base equal to a point, in that we can expand the category in which we can apply our methods.

2.

Birational Rigidity

In this chapter, we discuss the main definitions and Theorems of Birational rigidity, deriving them from a simpler definition, that of the *threshold of canonical adjunction* of a variety. These will form the basis behind the proof of Theorem 3.1.2 in Chapter 4. We will give a proof of the generalised $4n^2$-inequality, an improvement on the $4n^2$-inequality, a key ingredient in the proof of Theorem 3.1.2. Most of this material on Birational rigidity comes from the book [59], written and originally proved by Pukhlikov. Terminologically, we quickly remark that varieties in this section are assumed to be of arbitrary dimension unless indicated; the integer n is reserved for a part of the definition of a mobile linear system.

2.1. Main Definitions

2.1.1. The Threshold of Canonical Adjunction — We begin with the definition.

Definition 2.1.2. — Let X be a projective rationally connected variety with at worst $\mathbb{Q}$-factorial terminal singularities. The *threshold of canonical adjunction* of an effective divisor $D \subset X$ is the number $c(D, X) = \sup\{\epsilon \in \mathbb{Q}_+ \mid D + \epsilon K_X \in A^1_+ X\}$. If we let Σ be a non-empty linear system on X, then we similarly set $c(D, \Sigma) = c(D, X)$, where we take $D \in \Sigma$ to be arbitrary.

Theorem 2.1.3. — *For a variety X satisfying the above conditions, this number is finite.*

Proof. On a rationally connected variety, K_X is negative on at least one family of curves sweeping out X, whilst an effective divisor D is non-negative on such a family. Therefore, for $m \gg 0$, the linear system $|D + mK_X|$ is empty. $\square$

Remark 2.1.4. — From this point onward in this chapter, we assume that any given variety X satisfies the conditions given above, and hence always has a finite threshold canonical adjunction.

We work out some examples as follows:

Example 2.1.5. —

1. Let X be a *primitive* Fano variety, that is to say, suppose X is a variety with Picard group is generated by an ample anticanonical class, so that $\operatorname{Pic} X = \mathbb{Z}K_X$. For any effective divisor D, we have that $D \in |-nK_X|$ for some n, so that $c(D, X) = n$. Similarly, if we relax the primitivity condition to the case where $\operatorname{rk}\operatorname{Pic} X = 1$, so that $K_X = -rH$ where H is a hyperplane class which also generates the Picard group, and r is the index of the variety X, then for $D \in |nH|$ we get $c(D, X) = \frac{n}{r}$.

2. Let $\pi : X \to S$ be a rationally connected fibre space where $\dim X > \dim S \geq 1$, and let D_S be an effective divisor on the base. If we take the pullback of D_S to X, then we immediately see that $c(\pi^*(D_S), X) = 0$. If we further impose that X/S is a *standard rationally connected fibre space*, so that $\operatorname{Pic} X = \pi^* \operatorname{Pic} S \oplus \mathbb{Z}K_X$, and impose also that D is an effective divisor that isn't the pullback of a divisor on the base S, then $D \in |-nK_X + \pi^*R|$ for some divisor R on S and where $n \geq 1$. Clearly $c(D, X) \leq n$, whilst we have equality if the divisor R is effective.

Unfortunately, whilst this threshold is often very easy to compute, it is not a birational invariant and so not immediately helpful for us as the following example shows.

Example 2.1.6. — Let $\pi : \mathbb{P}^M \dashrightarrow \mathbb{P}^m$ be a linear projection from an $(M-m-1)$-dimensional plane $P \subset \mathbb{P}^M$. Let Σ_m be a mobile linear system of hypersurfaces of degree n in $\mathbb{P}^m$ and let Σ_M be its pullback on $\mathbb{P}^M$. We get that $c(\Sigma_M, \mathbb{P}^M) = \frac{n}{M+1}$. If we then blow up the plane P however, say $\sigma_P : \mathbb{P}^+ \to \mathbb{P}^M$, so that the composite map $\pi \circ \sigma : \mathbb{P}^+ \to \mathbb{P}^m$ is a $\mathbb{P}^{M-m}$-bundle, then in particular $\pi \circ \sigma$ is a morphism with rationally connected fibres. If we then let Σ^+ be the strict transform of Σ on $\mathbb{P}^+$, then by Example 2.1.5, we get $c(\Sigma^+, \mathbb{P}^+) = 0$.

Therefore, in order to overcome this non-invariance we define the following:

Definition 2.1.7. — Let Σ be a mobile linear system on a variety X. We define the *virtual threshold of canonical adjunction* by the formula

$$c_{\mathrm{virt}}(\Sigma) = \inf_{X^\sharp \to X} \{c(\Sigma^\sharp, X^\sharp)\}$$

where the infimum is taken over all birational morphisms $X^\sharp \to X$ where $X^\sharp$ is a non-singular projective model of $\mathbb{C}(X)$ and $\Sigma^\sharp$ is the strict transform of the system Σ on $X^\sharp$.

Clearly this is a birational invariant of the pair (X, Σ): if $\chi : X \to X'$ is a birational map, $\Sigma' = \chi_* \Sigma$ is the strict transform of the system Σ with respect to χ^{-1}, then we get $c_{\mathrm{virt}}(\Sigma) = c_{\mathrm{virt}}(\Sigma^+)$.

Proposition 2.1.8. —

1. *Assume that on a projective variety X there are no mobile linear systems with virtual threshold of canonical adjunction equal to 0. Then on X there are no structures of a non-trivial fibration into varieties of negative Kodaira dimension, that is to say, there is no rational dominant map $\rho : X \dashrightarrow S$, $\dim S \geq 1$, the generic fibre of which has negative Kodaira dimension.*

2. *Let $\pi : X \to S$ be a rationally connected fibre space. Assume that every mobile linear system Σ on X such that $c_{\mathrm{virt}}(\Sigma) = 0$ is the pullback of some mobile*

linear system Λ on S. Then any birational map

$$\begin{tikzcd} X \arrow[r, dashed, "\chi"] \arrow[d, "\pi"'] & X^\sharp \arrow[d, "\pi^\sharp"] \\ S & S^\sharp \end{tikzcd}$$

where $\pi^\sharp$ is a fibration into varieties of negative Kodaira dimension is fibrewise commutative, that is to say that there exists a rational dominant map $\rho : S \dashrightarrow S^\sharp$ making the diagram commutative. In other words, we can define an order on the set of rationally connected structures $RC(X)$ by setting $\pi^\sharp \geq \pi$: further π is the least element of $RC(X)$.

Proof. 1. Suppose we have such a fibration. Let Δ be a mobile linear system on S, and consider the system $\Sigma = \rho^*(\Delta)$. Then by Example 2.1.5 we get $c(X, \Sigma) = 0$; this contradicts our assumption.

2. Suppose there doesn't exist such a map ρ. This implies there exists an arbitrary mobile linear system $\Delta^\sharp$ on $S^\sharp$, whose pullback $\pi^{\sharp *}(\Delta^\sharp)$ satisfies $c_{\text{virt}}\pi^{\sharp *}(\Delta^\sharp)) > 0$. By hypothesis, this is a contradiction.

□

We can now state the main definitions of this section.

Definition 2.1.9. — 1. A variety X is said to be *birationally superrigid* if for any mobile linear system $\Sigma \subset |-nK_X|$ on X the following equality holds:

$$c_{\text{virt}}(\Sigma) = c(\Sigma, X).$$

2. A variety X (respectively, a rationally connected fibre space X/S) is said to be *birationally rigid* if for any mobile linear system Σ on X there exists a birational self-map $\chi \in \operatorname{Bir} X$ (respectively a fibrewise birational self-map $\chi \in \operatorname{Bir}(X/S)$) which gives the equality

$$c_{\text{virt}}(\Sigma) = c(\chi_*\Sigma, X).$$

This pair of definitions leads to the whole theory of Birational rigidity, one of the key methods to answering the rationality question for varieties.

Remark 2.1.10. — Note that a variety X being birationally superrigid immediately implies that is also birationally rigid; the converse does not hold in general. As an example, the intersection of a cubic and a quadric in $\mathbb{P}^5$ is rigid but not superrigid; see [37, Chapter 3] for a proof of this.

2.1.11. Contraction of Divisors — Therefore, supposing we wish to prove or disprove the Birational superrigidity of a rationally connected variety X, we begin by assuming by contradiction that there exists a mobile linear system Σ on X satisfying the inequality

$$c_{\text{virt}}(\Sigma) < c(\Sigma). \tag{2.1}$$

By definition, this implies that there exists a birational morphism

$$\phi : X^+ \to X$$

such that we have the inequality $c(\Sigma^+, X^+) < c(\Sigma)$, where Σ^+ is the strict transform of Σ. In particular, this implies the existence of at least one divisor $E \subset X^+$ which is contracted by the morphism ϕ (an irreducible component of the exceptional divisor of the map ϕ). Supposing that this weren't the case, that we had an isomorphism in codimension 1 (i.e. outside a closed subset $Y \subset X^+$) of codimension 2), then for any divisor $D \subset X$ and its strict transform D^+ we would have $c(D, X) = c(D^+, X^+)$ which contradicts our initial assumption.

Our divisor E determines a discrete valuation on the field of rational functions $\mathbb{C}(X)$, that is a function $\text{ord}_E : \mathbb{C}(X) \to \mathbb{Z} \cup \{\infty\}$ such that

- $\text{ord}_E(f \cdot g) = \text{ord}_E(f) + \text{ord}_E(g)$
- $\text{ord}_E(f + g) \geq \min\{\text{ord}_E(f), \text{ord}_E(g)\}$
- $\text{ord}_E(f) = \infty \iff f = 0.$

Note that this is independent of the choice of model X^+ in the following way; suppose we have another birational morphism $\phi^\sharp : X^\sharp \to X$ such that the birational map $(\phi^\sharp)^{-1} \circ \phi : X^+ \dashrightarrow X^\sharp$ is an isomorphism at a general point of the divisor E, so that $(\phi^\sharp)^{-1} \circ \phi(E) = E^\sharp \subset X^\sharp$ is an exceptional divisor of the morphism $\phi^\sharp$, then $\mathrm{ord}_E = \mathrm{ord}_{E^\sharp}$.

Remark 2.1.12. — Note that the irreducible subvariety $\phi(E) \subset X$ the centre of the discrete valuation ord_E as defined in Chapter 1 is independent of our choice of model.

In fact by applying the valuation ord_E to an effective divisor $D \subset X$ we obtain the *multiplicity* $\nu_E(D) \in \mathbb{Z}_+$ - we do this by looking at local equations, possible since our variety X is $\mathbb{Q}$-factorial. If we let $\mathcal{E}$ be the set of exceptional divisors of the birational morphism ϕ, then we get

$$\phi^* D = D^+ + \sum_{E \in \mathcal{E}} \nu_E(D) E. \tag{2.2}$$

Similarly, for the canonical class K_{X^+} we get

$$K_{X^+} = \phi^* K_X + \sum_{E \in \mathcal{E}} a(E) E, \tag{2.3}$$

where $a(E) = a(E, X) \geq 1$ is the *discrepancy* of the geometric valuation E, also independent of the model X^+.

Returning to our original setup, by assumption we have that $n = c(\Sigma) > 0$.

Definition 2.1.13. — A geometric discrete valuation ord_E of the field $\mathbb{C}(X)$ is called a *maximal singularity* of the linear system Σ if the *Noether-Fano* inequality

$$\nu_E(\Sigma) > na(E)$$

holds, where $\nu_E(\Sigma) = \nu_E(D)$ for a general divisor $D \in \Sigma$. In particular, we say that an irreducible subvariety $Y \subset X$ of codimension ≥ 2 is called a *maximal subvariety* of the linear system Σ if the inequality

$$\operatorname{mult}_Y \Sigma > n(\operatorname{codim} Y - 1)$$

holds, where $\operatorname{mult}_Y \Sigma = \operatorname{mult}_Y D$ for a general divisor $D \in \Sigma$.

Proposition 2.1.14. — *Assume that the inequality 2.1 holds. Then the linear system Σ has a maximal singularity.*

Proof. Let $\phi : X^+ \to X$ be a birational morphism from a non-singular variety X^+ satisfying the inequality $c(\Sigma^+) < c(\Sigma) = n$, $\mathcal{E}$ the set of divisors contracted by the morphism ϕ, $D \in \Sigma$ a general divisor, and let $D^+ \in \Sigma^+$ be its strict transform on X^+. From equations 2.2 and 2.3 we get

$$D^+ + nK_{X^+} = \phi^*(D + nK_X) - \sum_{E \in \mathcal{E}} e(E)E \notin A^1_+X$$

where $e(E) = \nu_E(D) - na(E)$ and the last non-inclusion holds by assumption. Since $D + nK_X \in A^1_+X$, and the pullback of a pseudoeffective class is necessarily pseudo-effective, we obtain that there exists at least one divisor E for which $e(E) > 0$. $\square$

Remark 2.1.15. — Note that we can reformulate the Noether-Fano inequality in terms of the language of $\mathbb{Q}$-divisors as follows. Let $D \in \Sigma$ be a general divisor. Then the Noether-Fano inequality states that the log pair $(X, \frac{1}{n}D)$ is not canonical, that is, has a non-canonical singularity $E \subset X^+$ satisfying the inequality $\nu_E(\frac{1}{n}D) > a(E)$.

We prove the following proposition, which is the most important implication of rigidity and superrigidity.

Proposition 2.1.16. — *Let X be a primitive Fano variety, X' a Fano variety with $\mathbb{Q}$-factorial terminal singularites and Picard number one (so that $\operatorname{Pic} X' \otimes \mathbb{Q} = \mathbb{Q}K'_X$) and let $\chi : X \dashrightarrow X'$ be a birational map.*

1. *Assume X is birationally rigid. Then X and X' are biregularly isomorphic, though χ is not necessarily an isomorphism.*

2. *Assume that X is birationally superrigid. Then χ is a biregular isomorphism, and in particular* $\operatorname{Bir} X = \operatorname{Aut} X$.

Proof. 1. Let $\chi : X \dashrightarrow X'$ be a birational map and $\phi : \tilde{X} \to X$ a log resolution, so that $\psi = \chi \circ \phi$ is a birational morphism. The variety $\tilde{X}$ is non-singular and

$$\operatorname{Pic} \tilde{X} = \mathbb{Z}\phi^* K_X \oplus \bigoplus_{E_i \in \mathcal{E}} \mathbb{Z}E_i$$

where $\mathcal{E}$ is the set of all the ϕ-exceptional divisors. By assumption

$$\operatorname{Pic} \tilde{X} \otimes \mathbb{Q} = \mathbb{Q}\psi^* K_{X'} \oplus \bigoplus_{E'_i \in \mathcal{E}'} \mathbb{Q}E'_i$$

where $\mathcal{E}'$ is the set of all the ψ-exceptional divisors. Set $K = \phi^* K_X, K' = \psi^* K_{X'}$. We get

$$K_{\tilde{X}} = K + \sum_{E_i \in \mathcal{E}} a_i E_i = K' + \sum_{E'_i \in \mathcal{E}'} a'_i E'_i \tag{2.4}$$

where $a_i \in \mathbb{Z}$, $a_i \geq 1$ and $a'_i \in \mathbb{Q}$, $a'_i > 0$.

Let $\Sigma' = |-mK_{X'}|$, $m \gg 0$ be a very ample linear system. Clearly $c(\Sigma', X') = m$. Taking its strict transform we get $\Sigma = \chi_*^{-1}\Sigma' \subset |-nK_X|$ for some n - similarly $c(\Sigma, X) = n$. By twisting with a suitable birational map (*) and the rigidity of X we may assume that we have equality of both virtual and actual thresholds for χ, so it follows that $n \leq m$. The strict transform of the linear system Σ on $\tilde{X}$ coincides with the strict transform of the linear system Σ' with respect to ψ. Therefore there exist positive integers b_i such that

$$-mK' = -nK - \sum_{E_i \in \mathcal{E}} b_i E_i.$$

and $b_i \leq m - n$ for every i. Dividing by $-m$ and substituting into the equation 2.4, we get

$$\left(1 - \frac{n}{m}\right) K = \sum_{E_i \in \mathcal{E}} \left(\frac{b_i}{m} - a_i\right) E_i + \sum_{E'_i \in \mathcal{E}'} a'_i E'_i.$$

Since the divisors E_i are ϕ-exceptional and $a'_i > 0$ for every i, we get the equality $n = m$. Furthermore, all the divisors E'_j turn out to be ϕ-exceptional and moreover $\mathcal{E} = \mathcal{E}'$, otherwise $\operatorname{rk}\operatorname{Pic} X' \geq 2$. Thus χ is an isomorphism in codimension one; set

$$U = X \setminus \bigcup_{E_i \in \mathcal{E}} \phi(E_i), \quad U' = X' \setminus \bigcup_{E'_i \in \mathcal{E}'} \psi(E'_i).$$

Then by the above $\chi : U \to U'$ is an isomorphism. Therefore $\Sigma = |-nK_X|$ and χ induces an isomorphism $\chi \circ \chi^*$ of the (ample) linear systems Σ and Σ', where χ^* is some map in $\operatorname{Bir} X$. Consequently, we can conclude that $\chi : X \to X'$ is an isomorphism.

2. This follows since we now no longer need to twist by a suitable birational map at (*).

□

This proposition presents the most important implications of the definition of rigidity and superrigidity. In particular, it is clear that rational varieties are neither rigid nor superrigid.

Remark 2.1.17. — There is an alternate definition of Birational (super)rigidity that applies to the category of Mori fibre spaces with morphisms given by birational maps, the so-called *Sarkisov Category*. These are end points of MMP applied to rationally connected varieties. We give it as follows, in a commonly seen form.

Definition 2.1.18. — [17, Definition 1.3] let $X \to Z$ and $X' \to Z'$ be Mori fibre spaces. A birational map $f : X \dashrightarrow X'$ is *square* if it fits into a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d] & X' \arrow[d] \\ Z \arrow[r, dashed, "g"] & Z' \end{tikzcd}$$

where g is birational and, in addition, the map $f_L : X_L \dashrightarrow X'_L$ induced on generic fibres is biregular, where L denotes a generic point of Z. In this case we say X/Z

and X'/Z' are *square birational*. A square birational map that is also biregular is called *square biregular*.

Definition 2.1.19. — We define the *pliability* of X to be the set

$$\mathcal{P}(X) = \{\text{Mori fibre spaces } Y \to T \sim \text{square equivalence}\}.$$

We say that X is *birationally rigid* if $\mathcal{P}(X)$ consists of a single element. Further, X is birationally superrigid if $\mathrm{Bir}(X) = \mathrm{Aut}(X)$.

Our definition in the absolute case covers the situation where the base is a point - in addition there is a slight asymmetry whereby X' is allowed to have torsion in the Picard group. However, in the study of varieties having undergone MMP we can safely ignore this distinction, so many authors prefer to take this simpler definition of Birational rigidity.

2.2. The Method of Maximal Singularities

We now describe the method of maximal singularities. This is the main method by which we prove Birational rigidity of varieties. We ask for a given geometric valuation ν_E of $\mathbb{C}(X)$ whether there exists a mobile linear system with a threshold of canonical adjunction $n = c(\Sigma) > 0$ for which ν_E is a maximal singularity. We then have two possibilities:

1. The answer is positive. Then we attempt to *untwist* the singularity E, that is to find a birational self map $\chi_E \in \mathrm{Bir}(X)$ such that $c({\chi_E^{-1}}_*\Sigma, X) < c(\Sigma, X)$. In this case, we can hope that our variety X is birationally rigid, though not superrigid. If it is not possible to untwist the singularity, then the variety is neither.

2. The answer is negative for any choice of geometric evaluation ν_E. In this case, the variety in question X is birationally superrigid.

Untwisting maps have been used successfully in many papers to prove the Birational rigidity of varieties. However, we choose to focus on what is known as *exclusion* of maximal singularities. This is where we suppose that the variety in study has such a singularity, and derive a contradiction based on this assumption.

In order to analyse a possible maximal singularity, the centre of which may well be *infinitely near* (which we now define), we use an associated *resolution* of the singularity. In approximate terms, we blow up successive centres of the contraction of the singularity until we arrive at a step where the contraction is a blow up. In concrete terms, we have the following:

Let X be a projective variety, and let $E \subset X^+$ be a divisor contracting to a centre $B \subset X$ by a birational map $\chi : X^+ \dashrightarrow X$ where $\operatorname{codim} B \geq 2$, with the condition that B is not contained in the singular locus of X. Let $\sigma_B : X(B) \to X$ be the blow up of the centre B with exceptional divisor $E(B) = \sigma_B^{-1}(B)$.

Proposition 2.2.1. — *1. One of the following holds: either the composition of birational maps $\sigma_B^{-1} \circ \psi : X^+ \dashrightarrow X$ is an isomorphism in a neighbourhood of the generic point of E, and in this case $\sigma_B^{-1} \circ \psi(E) = E(B)$, or $B^+ = \sigma_B^{-1} \circ \psi(E)$ is an irreducible subvariety of codimension greater than or equal to 2.*

2. Moreover, $B^+ \not\subset \operatorname{Sing} X(B)$, $B^+ \subset E(B)$ and $\sigma_B(B^+) = B$.

Proof. The first part is true by definition. To see the second part, note that $X(B)$ is non-singular outside the σ_B-preimage of the set $\operatorname{Sing} X \cup \operatorname{Sing} B$, $\sigma_B \circ \sigma_B^{-1} \circ \psi(E) = B$. $\square$

Remark 2.2.2. — Outside the preimage of the set $\operatorname{Sing} X \cup \operatorname{Sing} B$, we can see that the morphism $\sigma_B : E(B) \to B$ is a locally trivial $\mathbb{P}^{\operatorname{codim} B - 1}$-fibration and the discrepancy of the exceptional divisor $E(B)$ is $\operatorname{codim} B - 1$; this is a consequence of [33, Chapter 2, Exercise 8.5].

By repeatedly applying the above proposition we obtain a sequence of blow ups

$$\phi_{i,i-1} : \qquad \begin{array}{ccc} X_i & \longrightarrow & X_{i-1} \\ \cup & & \cup \\ E_i & \longrightarrow & B_{i-1} \end{array} \tag{2.5}$$

where $i \in \{1, \dots K\}$, $X_0 = X$, $B_0 = \text{centre}(E, X)$, B_j is the centre of E on X_j, and $E_i = \phi_{i,i-1}^{-1}(B_{i-1})$. In other words, we are successively blowing up the centres of the valuation E. The varieties $X_1, X_2, \dots$ can generally speaking be singular. However, each X_j is non-singular at the generic point of the subvariety $B_j \subset E_j$. For $i > j$ we set

$$\phi_{i,j} = \phi_{j+1,j} \circ \dots \circ \phi_{i,i-1} : X_i \to X_j$$

along with $\phi_{i,i} = id_{X_i}$.

In particular, by the previous proposition we see that $\phi_{i,j}(B_i) = B_j$ for $i > j$. For an irreducible subvariety $Y \subset X_j$ we denote its strict transform on X_i (supposing that Y $\not\subset B_j$ so that it is well-defined) by $Y^i \subset X_i$, adding the index i. We also use the same notation for effective algebraic cycles - for example if $Z = \sum m_k Z_k$ is a cycle on X_j, then its strict transform on X_i is given by $Z^i = \sum m_i Z_k^i$.

Proposition 2.2.3. — *The sequence of blow ups 2.5 terminates: that is to say that for some $K \geq 1$ the first case of Proposition 2.2.1 occurs, i.e. $\sigma_{K,0}^{-1} \circ \psi(E) = E_K$.*

Proof. We will see below that the discrepancies of the exceptional divisors E_i with respect to the model X will strictly increase; in particular, $a(E_i, X) \geq i$. At the same time, $a(E_i, X) \leq a(E, X)$ since the centre of E on X_i is contained in E_i. $\square$

Remark 2.2.4. — The sequence 2.5 is called the *resolution* of the discrete valuation ν_E with respect to the model X. On the set of exceptional divisors $\{E_1, \dots, E_K\}$ we introduce a structure of an oriented graph in the following way: the vertices E_i and E_j are joined by an oriented edge denoted by $i \to$ if $i > j$ and $B_{i-1} \subset E_j^{i-1}$. The graph structure formalises the operation of computing the strict transforms of exceptional divisors:

$$E_j^i = \phi_{i,j}^* E_j - \sum_{j \leftarrow k \leq i} \phi_{i,k}^* E_k$$

In order to compute the pullback in terms of the various strict transforms involved, set for $i > j$, p_{ij} to be the number of paths from E_i to E_j in the oriented graph described above, and set $p_{ii} = 1$

Proposition 2.2.5. — *The following decomposition holds:*

$$\phi_{i,j}^* E_j = \sum_{k=j}^{i} p_{kj} E_k^i \tag{2.6}$$

Proof. This statement is proved by induction on $i \geq j$. If $i = j$, there is nothing to prove. If $i = j+1$, then $\phi_{j+1,j}^* E_j = E_j^{j+1}$, since $B_j \subset E_j$ and E_j is non-singular at the generic point of B_j. For $i \geq j+2$ we get:

$$\begin{aligned} \phi_{i,j}^* E_j &= \phi_{i,i-1}^* \left(\sum_{k=j}^{i-1} p_{kj} E_k^{i-1} \right) \\ &= \sum_{k=j}^{i-1} p_{kj} E_k^i + \left(\sum_{\substack{k=j \\ B_{i-1} \subset E_k^{i-1}}} p_{kj} \right) E_i \end{aligned}$$

We then use the following equality:

$$p_{ij} = \sum_{i \to k} p_{kj}$$

where the arrow under the sum means that we care only about the first arrow. □

Remark 2.2.6. — The p_{ij} encode the multiplicities and discrepancies of all the blow ups and pullbacks involved. If we let Σ^j be the strict transform of the linear system Σ on X_j, we then set $\nu_j = \operatorname{mult}_{B_{j-1}} \Sigma^{j-1}$ and $\beta_j = \operatorname{codim} B_{j-1} - 1$. We subsequently get the traditional form of the Noether-Fano inequality:

$$\nu_{E_k}(\Sigma) = \nu_E(\Sigma) = \sum_{i=1}^{K} p_{Ki} \nu_i, \quad a(E) = \sum_{i=1}^{K} p_{Ki} \beta_i$$

In particular, by looking at the inequality in this form we see that the discrepancies are strictly increasing as claimed. Setting $p_i = p_{Ki}$ we obtain for a maximal singularity the most useful form of the *Noether-Fano inequality*:

$$\sum_{i=1}^{K} p_i \nu_i > n \sum_{i=1}^{K} p_i \beta_i \tag{2.7}$$

2.3. The $4n^2$ Inequalities

From a resolution of a maximal singularity $E \subset X^+$ we have two different cases. In the first, we have equality of dimension of all the centres $B_0, \ldots, B_{K-1}$. In the second, we have $\dim B_0 < \dim B_{K-1}$; we call this the *infinitely near case.* Now suppose we are in the infinitely near case, so that we can no longer use the second part of Definition 2.1.13. In particular, $\operatorname{codim} B \geq 3$. We set $B = B_0$ and consider the *self-intersection* of the linear system $Z = (D_1 \cdot D_2)$, where $D_1, D_2 \in \Sigma$ are general divisors. Recall that $n = c(\Sigma) > 0$ is the threshold of canonical adjunction and the Noether-Fano inequality holds. From this we have the following theorem:

Theorem 2.3.1. — *[59, Chapter 2, Section 2.2] The following inequality holds:*

$$\operatorname{mult}_B Z > 4n^2.$$

Proof. Our strategy is to divide the resolution $\phi_{i,i-1} : X_i \to X_{i-1}$ into a *lower* and an *upper* part, corresponding respectively to the indices $i = 1, \ldots, L \leq K$ where $\operatorname{codim} B_{i-1} \geq 3$ and the indices $i = L+1, \ldots, K$, where $\operatorname{codim} B_{i-1} = 2$. It may well be that $L = K$ occurs and hence the upper part is empty.
Consider the general divisors D_1 and D_2 as above. We define a sequence of codimension 2 cycles on each X_i, setting inductively

$$\begin{aligned}
&D_1 \cdot D_2 = Z_0, \\
&D_1^1 \cdot D_2^1 = Z_0^1 + Z_1, \\
&\cdots \\
&D_1^i \cdot D_2^i = (D_1^{i-1} \cdot D_2^{i-1}) + Z_i, \\
&\cdots
\end{aligned}$$

where in each case we have $Z_i \subset E_i$. Therefore for any $i \leq L$ we get

$$D_1^i \cdot D_2^i = Z_0^i + Z_1^i + \ldots + Z_{i-1}^i + Z_i.$$

For any $j > i,\ j \leq L$, set

$$m_{i,j} = \mathrm{mult}_{B_{j-1}}(Z_i^{j-1})$$

where we extend our notion of multiplicity of an irreducible subvariety along a smaller subvariety to arbitrary cycles by linearity.

Set $d_i = \deg Z_i$. We get the following system of equalities:

$$\begin{aligned}
&\nu_i^2 + d_1 = m_{0,1}\\
&\nu_2^2 + d_2 = m_{0,2} + m_{1,2}\\
&\ldots\\
&\nu_i^2 + d_i = m_{0,i} + \ldots + m_{i-1,i}\\
&\ldots\\
&\nu_L^2 + d_L = m_{0,L} + \ldots + m_{L-1,L}.
\end{aligned}$$

We also have the inequality

$$d_L \geq \sum_{i=L+1}^{K} \nu_i^2 \deg[(\phi_{i-1,L})_* B_{i-1}] \geq \sum_{i=L+1}^{K} \nu_i^2.$$

by Lemma 1.3.16. We now need the following definition.

Definition 2.3.2. — We say that a function $a : \{1, \ldots, L\}$ is *compatible with the graph structure* if

$$a(i) \geq \sum_{j \to i} a(j)$$

for any $i = 1, \ldots, L$.

In particular, $a(i) = p_i$ is such a function; we omit the proof of this (though it is a very easy exercise to show this).

Proposition 2.3.3. — *Let $a()$ be a function compatible with the graph structure. Then*

$$\sum_{i=1}^{L} a(i)m_{0,i} \geq \sum_{i=1}^{L} a(i)\nu_i^2 + a(L)\sum_{i=L+1}^{K} \nu_i^2.$$

Proof. We need the following two lemmas.

Lemma 2.3.4. — *If $m_{i,j} > 0$, then $j \to i$.*

Proof. If $m_{i,j} > 0$, then some component of Z_i^{j-1} contains B_{j-1}, whilst $Z_i^{j-1} \subset E_i^{j-1}$. □

Lemma 2.3.5. — *For any $i \geq 1$, $j \leq L$ we have $m_{i,j} \leq d_i$.*

Proof. The cycles B_λ are non-singular at their generic points, whilst the maps $\phi_{\lambda,\mu} : B_\lambda \to B_\mu$ are surjective. This means that we can count multiplicities at generic points. If we then blow up at a generic point, taking into account that multiplicities are non-increasing with respect to blowing up a non-singular variety, we reduce to the case of a hypersurface in projective space. But this is obvious. □

We multiply the i-th equality of the system by $a(i)$ and take the sum. This gives us on the left hand side:

$$\sum_{j\geq i+1}^{i=L-1} a(j)m_{i,j} \leq d_i \sum_{j\to i} a(j) \leq a(i)d_i.$$

using the above lemmas whilst on the right hand side we have

$$\begin{aligned} \sum_{i=1}^{L} a(i)\nu_i^2 + \sum_{i=1}^{L} a(i)d_i &\geq \sum_{i=1}^{L} a(i)\nu_i^2 + a(L)d_L \\ &\geq \sum_{i=1}^{L} a(i)\nu_i^2 + \sum_{i=L+1}^{K} a(L)\nu_i^2. \end{aligned}$$

Putting this all together gives the result. □

From this we get the following corollaries:

Corollary 2.3.6. — *Set* $m = m_{0,1} = \mathrm{mult}_B(D_1 \cdot D_2)$. *Then the following inequality holds:*

$$m\left(\sum_{i=1}^{L} a(i)\right) \geq \sum_{i=1}^{L} a(i)\nu_i^2 + a(L)\sum_{i=L+1}^{K} \nu_i^2.$$

Corollary 2.3.7. — *The following inequality holds:*

$$m\left(\sum_{i=1}^{L} p_i\right) \geq \sum_{i=1}^{K} p_i\nu_i^2.$$

Proof. For $i \geq L+1$ obviously $p_i \leq p_L$. □

Applying the Noether-Fano inequality, we can then minimise the right hand side of the corollary when

$$\nu_1 = \ldots = \nu_K = \frac{\sum_{i=1}^{K} p_i\beta_i n}{\sum_{i=1}^{K} p_i}.$$

If we now set

$$\Sigma_l = \sum_{\beta_j \geq 2} p_j, \quad \Sigma_u = \sum_{\beta_j = 1} p_j,$$

we get

$$\mathrm{mult}_B\, Z > \frac{(2\Sigma_l + \Sigma_u)^2}{\Sigma_l(\Sigma_l + \Sigma_u)} n^2.$$

It is then very easy to see that the right hand side is bounded below by $4n^2$, from which the result follows. □

This is the main tool we have to tackling the case of an infinitely near singularity. We have several refinements, two of which are mentioned in this thesis - the first is Lemma 2.3.8, which we use as an ingredient to proving the second, the *generalised* $4n^2$*-inequality*, and improves the bound in the case of singular points. We outline the proof below:

Lemma 2.3.8. — *Let* $o \in X$ *be a point on non-singular surface,* $C \ni o$ *a non-singular curve and let* Σ *be a mobile linear system on* X. *Let* $Z = (D_1 \cdot D_2)$ *be the self-intersection of the linear system* Σ, *an effective* 0*-cycle. We may assume that* *the cycle Z is concentrated at the point o. Assume that for a positive real number $a < 1$ and integer $n > 0$ the pair*

$$\left(X, \frac{1}{n}\Sigma + aC\right)$$

is not log canonical. Then the following inequality holds:

$$\deg Z > 4(1-a)n^2.$$

This is a special case of [16, Theorem 3.1] where a point on a normal crossings curve is considered. It was later extended by Mustaţă to the case where a can be arbitrarily positive.

From this we now prove the generalised $4n^2$-inequality, our setup is as follows: Let (X, o) be a complete intersection singularity of codimension l and type $\underline{\mu} = (\mu_1, \ldots, \mu_l)$, where

$$\dim X = M \geq l + \mu_1 + \ldots + \mu_l + 3.$$

We assume that the singularity is *generic*, to be defined below. Our theorem takes the form:

Theorem 2.3.9. — *[65, Theorem] Let Σ be a mobile linear system on X. Assume that for some positive $n \in \mathbb{Q}$ the log pair $(X, \frac{1}{n}\Sigma)$ is not canonical at the point o but canonical outside this point. Then the self-intersection $Z = (D_1 \cdot D_2)$ of the system Σ satisfies the inequality*

$$\operatorname{mult}_o Z > 4n^2 \operatorname{mult}_o X.$$

Remark 2.3.10. — In other words, this theorem gives us a very strong lower bound on the multiplicity at a centre of a maximal singularity of the self-intersection of a general pair of divisors in a linear system. This allows us to exclude centres which are singular points as well as non-singular - as will be seen this will be very useful for the main result of the thesis.

Proof. Our strategy is to reduce to the previous theorem by blowing up the point o under certain assumptions.

The germ (X, o) is given locally by a system of l algebraic equations

$$\begin{aligned} 0 &= q_{1,\mu_1} + q_{1,\mu_1+1} + \ldots + q_{1,d_1} \\ &\ldots \\ 0 &= q_{l,\mu_l} + q_{l,\mu_l+1} + \ldots + q_{l,d_l} \end{aligned}$$

in an affine chart $\mathbb{C}^{M+l}$, where the polynomials $q_{j,i}$ are homogeneous of degree i in the coordinates $z_1, \ldots, z_{M+l}$ and where at least one of the μ_i is greater than or equal to two; the point $o = (0, \ldots, 0)$ is the origin. We denote by

$$\underline{\mu} = (\mu_1, \ldots, \mu_l)$$

the type of the singularity $o \in X$ and set

$$\mu = \prod \mu_i = \operatorname{mult}_o X$$

to be the multiplicity of the point o. We also set

$$\left|\underline{\mu}\right| = \sum_i \mu_i.$$

We recall that by assumption $M \geq l + \left|\underline{\mu}\right| + 3$. Let $P \ni o$ be a linear subspace of $\mathbb{C}^{M+l}$ of dimension $2l + \left|\underline{\mu}\right| + 3$. We denote by X_P the intersection $X \cap P$.

Definition 2.3.11. — We say that the complete intersection singularity (X, o) is *generic*, if for a general subspace P of dimension $2l + \left|\underline{\mu}\right| + 3$ the singularity $o \in X_P$ is an isolated singularity, $\dim X_P = l + |\mu| + 3$ and for the blow up

$$\phi_P : X_P^+ \to X_P$$

of the point o, the variety X_P^+ is non-singular in a neighbourhood of the exceptional divisor $Q_P = \phi_P^{-1}(o)$, which is a non-singular complete intersection

$$Q_P = \{q_{1,\mu_1} = q_{2,\mu_2} = \ldots = q_{l,\mu_l} = 0\} \subset \mathbb{P}^{2l+\left|\underline{\mu}\right|+2}$$

of codimension l and type $\underline{\mu} = (\mu_1, \ldots, \mu_l)$.

From this point, assume that $o \in X$ is generic. In particular, by Theorem 1.2.6, X is a factorial variety near o. Let us begin the proof.
For a general $(2l + |\underline{\mu}| + 3)$-subspace P, set $\Sigma_P = \Sigma|_P$ to be the restriction of Σ onto P. By inversion of adjunction, proved below, the pair $(X_P, \frac{1}{n}\Sigma_P)$ is not canonical. We also have

$$Z_P = Z|_P = (Z \cdot X_P)$$

is the self-intersection of the system and $\operatorname{mult}_o Z = \operatorname{mult}_o Z_P$. Therefore, we may assume that $M = l + |\mu| + 3$ and so $P = \mathbb{C}^{M+l}$ from the start, so that our original singularity $o \in X$ is isolated. Therefore, we can omit the index P and hence write

$$\phi : X^+ \to X$$

for the blow up of the point o and $Q = \phi^{-1}(o)$ for the exceptional divisor, which as before is a non-singular complete intersection of codimension l and type $\underline{\mu}$ in $\mathbb{P}^{2l+|\mu|+2}$.
At this point we restric to a generic linear subspace $\Pi \ni o$ of dimension $|\underline{\mu}| + 3$. Let X_Π denote the intersection $X \cap \Pi$. Similarly, we let

$$\phi_\Pi : X_\Pi^+ \to X_\Pi$$

be the blow up of the point o and let $Q_\Pi = \phi^{-1}(o)$ be the exceptional divisor. In addition, by the adjunction formula we have the equality

$$a(Q_\Pi, X_\Pi) = 2.$$

If we now take a general divisor $D \in \Sigma$ and its strict transform $D^+ \in \Sigma^+$ on X^+ we have

$$D^+ \sim -\nu Q$$

for some positive integer ν. If $\nu > 2n$, then

$$\operatorname{mult}_o Z \geq \nu^2 \mu > 4\mu n^2$$

and so we have the desired inequality. Therefore, we assume the converse, that

$$\nu \leq 2n.$$

If we set $D_\Pi = D|_{X_\Pi}$, we get $D_\Pi^+ \sim -\nu Q_\Pi$. Again, by inversion of adjunction the pair $(X_\Pi, \frac{1}{n}D_\Pi)$ is not canonical at the point o, so for some exceptional divisor E_Π lying over X_Π the Noether-Fano inequality

$$\operatorname{ord}_{E_\Pi} \Sigma_\Pi > na(E_\Pi, X_\Pi)$$

holds. This implies that $E_\Pi \neq Q_\Pi$ since $\nu \leq 2n$ and $a(Q_\Pi, X_\Pi) = 2$, hence E_Π is a non canonical singularity of the pair

$$\left(X_\Pi^+, \frac{1}{n} D_\Pi^+ + \frac{\nu - 2n}{n} Q_\Pi \right).$$

Let $\Delta_\Pi \subset Q_\Pi$ denote the centre of E_Π on X_Π^+, which is an irreducible subvariety in Q_Π.

Proposition 2.3.12. — *Suppose that* $\operatorname{codim}(\Delta_\Pi \subset Q_\Pi) = 1$*, then the estimate*

$$\operatorname{mult}_o Z \geq 8n^2\mu$$

holds.

Proof. Begin by noting that by the genericity of the subspace Π, the multiplicity remains the same on restriction so that $\operatorname{mult}_o Z = \operatorname{mult}_o Z_\Pi$. Using Lemma 2.3.8, we get the following inequality:

$$\operatorname{mult}_o Z_\Pi \geq \nu^2\mu + 4\left(3 - \frac{\nu}{n}\right) n^2\mu$$

from which the claim follows. □

Remark 2.3.13. — This was first proved for the case of a non-singular point (in a slightly different setup) as [4, Lemma 5.3]

Therefore, returning to the proof of the theorem, we can assume that $\operatorname{codim}(\Delta_\Pi \subset Q_\Pi) \geq 2$. Therefore, returning to our original variety X, by taking into account the first blow up and the Noether-Fano inequality, we can conclude that for some exceptional divisor E lying over X we get the inequality

$$\operatorname{ord}_E \Sigma > n(2 \operatorname{ord}_E Q + a(E, X^+))$$

such that the centre $\Delta \subset Q$ of E on X has codimension at least 2 and dimension at least $2l$. At this point, as above, we can now resolve the singularity.
Consider, as in the first case, the resolution of the singularity E

$$X = X_0 \leftarrow X^+ = X_1 \leftarrow X_2 \leftarrow \ldots \leftarrow X_K,$$

with the same notation as in the previous section so that the blow ups are given by maps $\phi_{i,i-1} : X_i \to X_{i-1}$ with centres $B_{i-1} \subset X_{i-1}$ and exceptional divisors E_{i-1}. In this case however, we can already say that $B_0 = o$ and $B_1 = \Delta$, so that $E_1 = Q$. Since $X_1 = X^+$ is non-singular in a neighbourhood of E_1, so too are all subsequent varieties X_i non-singular at the generic point of B_i and hence we can use all the previous constructions automatically.

As before, recall that the last exceptional divisor E_K defines the discrete valuation ord_E. Similarly, divide the sequence $\phi_{i,i-1}$ into the *lower part* with indices $i = 1, \ldots, L$ and the *upper part* where $i = L+1, \ldots, K$. As before we also denote the strict transform of any geometric object on X_i by adding the upper index i and set:

$$\nu_i = \operatorname{mult}_{B_{i-1}} \Sigma^i$$

for any $i = 2, \ldots, K$. We now have an inequality of the form

$$\sum_{i=1}^{K} p_i \nu_i > \left(2p_1 + \sum_{i=2}^{K} p_i \beta_i \right) \tag{2.8}$$

where $\beta_i = \operatorname{codim}(B_{i-1} \subset X_{i-1})$ and $\nu_1 = \nu$. By linearity of inequality 2.8 and the standard properties of the numbers p_{ij} we may assume that $\nu_K > n$, replacing if necessary E_K by a lower singularity E_j for some $j < K$. By Theorem 1.3.18, applying the result to a divisor in the linear system $\Sigma^1|_Q$, we conclude that $\nu_1 \geq \nu_2$, since $\dim B_1 \geq 2l$. The inequalities

$$\nu_2 \geq \nu_3 \geq \ldots \geq \nu_K$$

hold as standard.
We now take a general pair of divisors D_1, D_2, as in the proof of the traditional $4n^2$-inequality and set

$$Z = Z_0 = (D_1 \circ D_2)$$

to be the self-intersection of the mobile linear system Σ. Again denote, where appropriate, by an upper index i the strict transform of some geometric object on X_i. For $i \geq 1$ we write

$$(D_1^i \circ D_2^i) = (D_1^{i-1} \circ D_2^{i-1})^i + Z_i,$$

where Z_i is supported on E_i, is a codimension 2 cycle, and hence may be seen as an effective divisor on E_i. Therefore, for any $i \leq L$ we obtain the presentation

$$(D_1^i \circ D_2^i) = Z_0^i + Z_1^i + \ldots Z_{i-1}^i + Z_i.$$

For the effective divisor Z_1 on $E_1 = Q$ (we can view it as such as Z_i is an effective codimension 2 cycle supported on E_i) we have the relation

$$Z_1 \sim d_1 H_Q$$

for some $d_1 \in \mathbb{Z}_+$, where H_Q is the class of a hyperplane section of the complete intersection $Q \subset \mathbb{P}^{4l+2}$. Once again we get a system of equalities

$$\begin{aligned}
&\mu(\nu_1^2 + d_1) = m_{0,1} \\
&\nu_2^2 + d_2 = m_{0,2} + m_{1,2} \\
&\ldots \\
&\nu_i^2 + d_i = m_{0,i} + \ldots + m_{i-1,i} \\
&\ldots \\
&\nu_L^2 + d_L = m_{0,L} + \ldots + m_{L-1,L}
\end{aligned}$$

where the multiplicities $m_{i,j}$ are defined as before and the estimate

$$d_L \geq \sum_{i=L+1}^{K} \nu_i^2$$

holds as usual. The theorem now follows from the following proposition:

Proposition 2.3.14. — *The following pair of inequalities holds:*

1. $d_1 \geq m_{1,2}$;

2. $m_{0,1} \geq \mu m_{0,2}$.

Proof. The first part follows from Theorem 1.3.18 as $Z_1 \sim d_1 H_1$ and $\dim B_1 \geq 2l$. For the second part, we note we have the numerical equivalence

$$\begin{aligned}
(Z^1 \cdot E_1) &\sim \frac{1}{\mu} \deg(Z^1 \cdot E_1) H_Q^2 \\
&\sim \frac{1}{\mu} m_{0,1} H_Q^2
\end{aligned}$$

as $m_{0,1} = \deg(Z^1 \cdot E_1)$. Applying Theorem 1.3.18 to the cycle $(Z \cdot Q)$, we get the inequality

$$m_{0,2} \leq \operatorname{mult}_\Delta(Z^1 \cdot Q) \leq \frac{1}{\mu} m_{0,1},$$

which completes the proof of the proposition. □

In fact, we can say further, that $m_{0,1} \geq \mu m_{0,1}$ for $i \geq 3$. If we then set

$$m^*_{i,j} = \mu m_{i,j}$$

for $(i,j) \neq (0,1)$ and $m^*_{0,1} = m_{0,1}$, as well as $d^*_i = \mu d_i$ for $i = 1, \ldots, L$, we obtain the following system of inequalities:

$$\begin{aligned}
&\mu\nu_1^2 + d_1^* = m^*_{0,1} \\
&\nu_2^2 + d_2^* = m^*_{0,2} + m^*_{1,2} \\
&\ldots \\
&\nu_i^2 + d_i^* = m_{0,i^*} + \ldots + m^*_{i-1,i} \\
&\ldots \\
&\nu_L^2 + d_L = m^*_{0,L} + \ldots + m^*_{L-1,L}
\end{aligned}$$

as well as

$$d^*_L \geq \mu \sum_{i=L+1}^{K} \nu_i^2$$

where the integers $m^*_{i,j}$ and d^*_i satisfy precisely the same properties as the integers $m_{i,j}$ and d_i in the non-singular case. Repeating the same arguments verbatim we obtain the inequality

$$\left(\sum_{i=1}^{L} p_i\right) \operatorname{mult}_o Z \geq \mu \sum_{i=1}^{K} p_i \nu_i^2$$

and by the same argument the desired inequality

$$\operatorname{mult}_o Z > 4\mu n^2.$$

$\square$

Remark 2.3.15. — Note that if we allow the case where $\mu_i = 1$ for every i, then this theorem reduces to the older $4n^2$-inequality as the point o automatically satisfies the genericity condition on the singularity. This justifies our terminology for the theorem.

Remark 2.3.16. — It would be lovely if we were able to relax the condition of genericity - in general however this very difficult, as elucidated by the following example. If we take a singular variety $S \subset \mathbb{P}^n$, and take the affine cone over S where the vertex is the point o, blowing up this point yields an exceptional divisor isomorphic to the original variety S, so we cannot reduce immediately to the non-singular case. In principle it should be possible to prove the theorem for individual varieties in certain geometric problems. However, in any case, due to the nature of proving Birational rigidity of higher dimensional varieties using linear systems introduces a degree of genericity anyway, we will not worry so much about this.

2.4. Inversion of Adjunction

As part of the proof of the generalised $4n^2$-inequality, we had to use the *inversion of adjunction*. This allows us to relate the discrepancies in the neighbourhood of a point $p \in X$ and the discrepancies of the same point on a subvariety $X \cap S$, where S is a hypersurface containing the point p. We call this method of simplifying some Birational rigidity type arguments the *linear method*. Its proof follows by blowing up at the point in question and using the following theorem, known as *the connectedness principle*; we recall the proof given for [40, Theorem 7.4].

Theorem 2.4.1. — *Let X be a normal variety and let $D = \sum d_i D_i$ be an effective $\mathbb{Q}$-divisor on X such that $(K_X + D)$ is a $\mathbb{Q}$-Cartier divisor. Let $f : Y \to X$ be a log resolution of the pair (X, D). Define*

$$K_Y \equiv f^*(K_X + D) + \sum_{i \in I} e_i E_i$$

and let $A = \sum_{e_i > -1} e_i E_i$ and $F = -\sum_{e_i \leq -1} e_i E_i$. Then $\mathrm{Supp}(F) = \mathrm{Supp}(\lfloor F \rfloor)$ *is connected in a neighbourhood of any fibre of F.*

Proof. We begin by noting that

$$\lceil A \rceil - \lfloor F \rfloor = K_Y + (-f^*(K_X + D)) + \{-A\} + \{F\}.$$

By the Kawamata-Viehweg vanishing theorem we have the vanishing of the right derived groups $R^1 f_* \mathcal{O}_Y(\lceil A \rceil - \lfloor F \rfloor) = 0$. Applying f_* to the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_Y(\lceil A \rceil - \lfloor F \rfloor) \longrightarrow \mathcal{O}_Y(\lceil A \rceil) \longrightarrow \mathcal{O}_{\lfloor F \rfloor}(\lceil A \rceil) \longrightarrow 0$$

we see that the map $f_* \mathcal{O}_Y(\lceil A \rceil) \to f_* \mathcal{O}_Y \lfloor F \rfloor(\lceil A \rceil)$ is surjective.

Let E_j be an irreducible component of A, so that E_j is either an exceptional divisor of the strict transform of some D with $d_i < 1$ - this implies that the divisor $\lceil A \rceil$ is completely exceptional, and hence $f_* \mathcal{O}_Y(\lceil A \rceil) = \mathcal{O}_X$. Suppose by contradiction that $\lfloor F \rfloor$ had at least two connected components, so that $\lfloor F \rfloor = F_1 \sqcup F_2$ in a neighbourhood of $f^{-1}(x)$ for some point $x \in X$. Then the stalks

$$f_* \mathcal{O}_{\lfloor F \rfloor}(\lceil A \rceil)_{(x)} \cong f_* \mathcal{O}_{F_1}(\lceil A \rceil)_{(x)} \oplus f_* \mathcal{O}_{F_2}(\lceil A \rceil)_{(x)}$$

and both summands are necessarily non-zero. Therefore $f_* \mathcal{O}_{\lfloor F \rfloor}(\lceil A \rceil)_{(x)}$ cannot be a quotient of a module generated by a single element. In particular, $\mathcal{O}_{X,x} \cong f_* \mathcal{O}_Y(\lceil A \rceil)_{(x)}$ is such a module, a contradiction, since $f_* \mathcal{O}_{\lfloor F \rfloor}(\lceil A \rceil)_{(x)}$ is clearly a quotient of this module. $\square$

Now that we have the connectedness principle available to us, we are in a position to prove the inversion of adjunction.

Theorem 2.4.2. — *Let $o \in X$ be a point on a $\mathbb{Q}$-factorial terminal variety, and $D \subset X$ an effective $\mathbb{Q}$-divisor, the support of which contains o. Let $R \subset X$ be an Cartier divisor where $o \ni R \not\subset \operatorname{Supp} D$. Assume that the pair (X, D) is not canonical at the point o, but canonical outside that point. Then the pair $(R, D_R = D|_R)$ is not log canonical at the point o.*

Remark 2.4.3. — When we say that (X, D) is not canonical at a point o but canonical outside that point, we mean that any exceptional divisors which appear with a negative coefficient in a resolution of X necessarily map to the point x under the resolution morphism.

Proof. Let $D = \sum_{i \in I} d_i D_i$ be an effective $\mathbb{Q}$-divisor, so that $d_i \in \mathbb{Q}_+$ for every $i \in I$. Since the pair (X, D) is canonical outside the point o, we get the inequality $d_i \leq 1$ for all $i \in I$. Further, we can assume that every $d_i < 1$ be replacing D by $\frac{1}{1+\epsilon} D$ for a small value of $\epsilon \in \mathbb{Q}_+$.

Let $\phi : \tilde{X} \to X$ be a resolution of singularities of the pair $(X, D + R)$. We write

$$K_{\tilde{X}} = \phi^*(K_X + D + R) + \sum_{j \in J} e_j E_j = \sum_{i \in I} d_i \tilde{D}_i - \tilde{R}, \tag{2.9}$$

where the exceptional divisors of the morphism ϕ are all the E_j, and $\tilde{D}_i$ and $\tilde{R}$ are the strict transforms of the divisors D_i and R on $\tilde{X}$ respectively. Set

$$b_j = \operatorname{ord}_{E_j} \phi^* D, \quad a_j = a(E_j, X)$$

for every $j \in J$. From this we get that $e_j = a_j - b_j - r_j$, where $r_j = \operatorname{ord}_{E_j} \phi^* R$. If we then consider the pullback of the point o we obtain

$$\phi^{-1}(x) = \bigcup_{j \in J^+} E_j$$

for some subset $J^+ \subset J$. Recalling that R is some Cartier divisor on X containing the point o, we get that for $j \in J^+$,

$$r_j = \operatorname{ord}_{E_j} \phi^* R \geq 1.$$

Further, since the pair (X, D) is not canonical at o, but is canonical outside that point, there exists among the indices $j \in J^+$ an index k such that $a_k < b_k$. For this index we have $e_k < -1$ corresponding to an exceptional divisor E_k. In particular, by the connectedness principle we have

$$E_k \cap \tilde{R} \neq \emptyset.$$

We now apply the adjunction formula to get

$$K_{\tilde{R}} = (K_{\tilde{X}} + \tilde{R})|_{\tilde{R}} = \phi_R^*(K_R + D_R) + \left(\sum_{j\in J} e_j E_j|_{\tilde{R}} - \sum_{i\in I} d_i \tilde{D}_i|_{\tilde{R}}\right)$$

where $\phi_R = \phi|_{\tilde{R}} : \tilde{R} \to R$ is the restriction of the resolution map ϕ onto R. The coefficient of E_k is then strictly less than -1. $\square$

2.5. Hypertangent Divisors

In this section we give a description of the technique of hypertangent divisors. This was first used in the paper [60] where it was proved that a general hypersurface of degree d embedded in $\mathbb{P}^d$ where $d \geq 5$ is birationally superrigid, and has been used successfully many times since then for numerous classes of families. It also has applications in the calculation of canonical thresholds, which we will discuss in Chapter 4.

Definition 2.5.1. — Let X be a variety, and let $\pi : X^+ \to X$ be the blow up of an arbitrary point $o \in X$. Assume that the exceptional divisor $E = \pi^{-1}(o)$ is reduced and irreducible. An effective divisor on X is said to be *hypertangent* to X (with respect to a point o) if the strict transform D^+ of the divisor D is an element of the linear system $|kH - lE|$ for some $l \geq k+1$. The number $\beta(D) = \frac{l}{k}$ is then called the *slope* of the divisor.

The most important fact about hypertangent divisors is the following:

Lemma 2.5.2. — *Let D be a hypertangent divisor on a variety X in the linear system $|kH - lE|$, where $l \geq k+1$, with slope $\beta(D) = \frac{l}{k}$. Then for any irreducible subvariety Y of X such that $Y \not\subset |D|$, where $|D|$ is the set of points defined by the divisor D, the algebraic cycle equal to the scheme-theoretic intersection $(D \circ Y)$ satisfies the following:*

$$\frac{\mathrm{mult}_o}{\deg}(Y \circ D) \geq \beta(D) \frac{\mathrm{mult}_o}{\deg} Y.$$

This more or less directly follows on from the definition of hypertangent divisor, yet is the main tool by which we are able to prove the Birational superrigidity of the varieties of interest. This works well for hypersurfaces, but runs into trouble with complete intersections due to the containment condition. Beginning in the paper [53], the following generalisation of this method was used that can tackle this problem.

Definition 2.5.3. — Let $\pi : X^+ \to X$ be the blow-up of X at the point o. A non-empty linear system Σ on X is said to be *hypertangent* (with respect to the point o) if $\Sigma^+ \subset |kH - lE|$, where E is the exceptional divisor, l and k are positive integers such that $l \geq k+1$, and Σ^+ is the strict transform of the system Σ on X^+. The number $\beta(\Sigma) = \frac{l}{k} > 1$ is called the *slope* of the system Σ.

Rather usefully it is the case that a set of hypertangent divisors $\mathcal{D}$ generates a hypertangent linear system $\Sigma_k = \Sigma_k(\mathcal{D})$ in the following way: for each $D \in \mathcal{D}$, define k_D and l_D to be the coefficients in the expression of the strict transform $D^+ \in |k_D H - l_D E|$. Let

$$f_D \in H^0(X, \mathcal{O}_X(k_D H))$$

be a section defining the divisor D. Set

$$\Sigma_k = \left| \sum_{k_D \leq k} f_D s_D = 0 \right|$$

where the summation is taken over all hypertangent divisors $D \in \mathcal{D}$ such that $k_D \leq k$, and s_D is an arbitrary polynomial of degree $(k - k_D)$ with a zero of order $(k - k_D)$ at the point o. Clearly

$$\beta(\Sigma_k) \geq \min_{D \in \mathcal{D},\, k_d \leq k} \left\{ \frac{k + l_D - k_D}{k} \right\}.$$

In fact, in most cases and in the case considered in Chapter 4, for every $D \in \mathcal{D}$ we have that $l_D = k_D + 1$, so that all the slopes are of the form

$$\beta(\Sigma_k) \geq \frac{k+1}{k}.$$

Further, we get the equality

$$\operatorname{codim}_o \operatorname{Bs} \Sigma_k = \#\{D \in \mathcal{D} \mid k_D \leq k\}.$$

Define the *ordering function*

$$\chi : \{1, \ldots, N\} \to \mathcal{K} = \{k_D \mid D \in \mathcal{D}\}$$

by the relation

$$\#\{D \in \mathcal{D} \mid k_D < \chi(i)\} < i \leq \#\{D \in \mathcal{D} \mid k_D \leq \chi(i)\}. \tag{2.10}$$

For example,

$$\chi(1) = \min\{k_D \mid D \in \mathcal{D}\}, \quad \chi(N) = \max\{k_D \mid D \in \mathcal{D}\}.$$

By construction, we finally obtain

$$\operatorname{codim}_o \operatorname{Bs} \Sigma_{\chi(i)} \geq i.$$

Using this technique, as well as Lemma 2.5.2, which directly carries over to this environment, we are able to prove the Birational superrigidity of higher dimensional varieties. We do this by constructing a sequence of varieties, essentially by successive intersection to derive a contradiction; we will show how to do this as an example in the proof of Theorem 3.1.2.

3.

Cyclic Covers

In this chapter we prove the main result of the paper [29], the Birational superrigidity of a general cyclic cover of a Fano hypersurface with an isolated singular point of high multiplicity of index one where the singular point does not lie on the ramification divisor. We use the term "high" in this case to distinguish from other applications of the theory of hypertangent divisors where we restrict ourselves to the case of quadratic singularities (of bounded rank).

3.1. Introduction

3.1.1. Statement of the main result. — Let $M \geq 6$, and let $G = G_m \subset \mathbb{P}^{M+1} = \mathbb{P}$ be a hypersurface of degree m containing a single isolated singular point o with multiplicity μ, where $\mu < M - 4$. We then let

$$\sigma : F \to G$$

be a $K : 1$ cyclic cover branched over a divisor $W \cap G$ where $W = W_{Kl} \subset \mathbb{P}$ is a hypersurface of degree Kl and $o \notin W$. Introducing a new coordinate u of weight l, we can realise F as a complete intersection of type $m \cdot Kl$ in the weighted projective space

$$\mathbb{P}^* = \mathbb{P}(\underbrace{1, \ldots, 1}_{M+2}, l).$$

Namely, F is given by the system of equations

$$f(x_0, \ldots, x_{M+1}) = 0, \quad u^K = g(x_0, \ldots, x_{M+1})$$

where f and g are homogeneous polynomials of degrees m and Kl respectively - f corresponds to the hypersurface G whilst g corresponds to the branch divisor W.

We further require that the polynomials f and g satisfy some regularity conditions at every point $p \in F$, stated in Section 3.2. Since o is a singular point and $M \geq 6$, the variety remains factorial by Theorem 1.2.6. Then, using Theorem 1.2.9, we impose on the integers m, l and K that they satisfy the relation $m + (K-1)l = M + 1$. This means that F is a primitive Fano variety of dimension M, which is to say that $\operatorname{Pic} F = \mathbb{Z}K_F$ and $(-K_F)$ is ample.

We also assume that

$$(Kl)^2 - 5Kl + 10 \geq 2m. \tag{3.1}$$

The proof that the codimension of the subspace of the defining parameter space where the regularity conditions fail is positive requires this inequality, and is compatible with the previous choice of paramaters of m, K and l.
The theorem is then:

Theorem 3.1.2. — *A general (in the Zariski topology) variety F of the type described above is birationally superrigid. In particular, F admits no non-trivial structures of a rationally connected fibration, any birational map $F \dashrightarrow F^\sharp$ onto a Fano variety with $\mathbb{Q}$-factorial terminal singularities and whose Picard group satisfies* $\operatorname{rk}\operatorname{Pic} F^\sharp = 1$ *is an isomorphism, and further the groups of birational and biregular self-maps coincide:*

$$\operatorname{Bir} F = \operatorname{Aut} F.$$

When we say "general" in the statement of the theorem above, we mean that the defining polynomials f and g satisfy some regularity conditions whose failure implies that the pair of polynomials f and g belong to a strict closed subset of the defining parameter space. We explain precisely what we mean in the coming section.

3.2. The Regularity Conditions

Since F is determined by two polynomials f and g of degrees m and Kl respectively, we can view F as a point $\underline{f}$ in the parameter space $\mathcal{F}$

$$\underline{f} \in \mathcal{F} \subset H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m)) \times H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(Kl))$$

under the following conditions for $\mathcal{F}$:

- for a pair of polynomials $(f, g) = \underline{f} \in \mathcal{F}$, the corresponding Fano cyclic cover $F = \mathbb{V}(f, g) \subset \mathbb{P}^*$ is irreducible and reduced.
- deconstructing the local equation for f at the point o (here we view f as defining the variety G) into homogeneous components, the initial $\mu - 1$ components all vanish, that is to say locally

  $$f = q_\mu + q_{\mu+1} + \ldots + q_m,$$

  where q_i is the ith homogeneous component of f and q_μ is not identically zero, whilst at every other point o', the equation is locally given in homogeneous components as

  $$f = q'_1 + q'_2 + \ldots + q'_m,$$

  where again q_1 is not identically zero.
- the points $\sigma^{-1}(o)$ are not contained in the variety defined by the equation $g = 0$.
- the variety defined by the vanishing of the polynomial $u^K = g$ is non-singular.

This set $\mathcal{F}$ makes a natural parameter space for Fano cyclic covers in question.

We also need a condition on the blow up of the singular point $o \in G$:

(R0.1) **Condition on the singularity on the base.** Let $\phi_{\mathbb{P}} : \mathbb{P}^+ \to \mathbb{P}$ be the blow up of the point o on G, $E_{\mathbb{P}} = \phi_{\mathbb{P}}^{-1}(o) \cong \mathbb{P}^M$ the exceptional divisor, $G^+ \subset \mathbb{P}^+$ the strict transform of the hypersurface G, so that $\phi : G^+ \to G$ is the blow up of the point o on G and $E = G^+ \cap E_{\mathbb{P}}$ is the exceptional divisor. We require that the subvariety

$$E \subset E_{\mathbb{P}} \cong \mathbb{P}^M$$

is a non-singular hypersurface in its linear span, i.e.

$$\langle E \rangle \cong \mathbb{P}^M.$$

In other words, supposing $(z_0, z_1, \ldots, z_{M+1})$ is a set of affine coordinates at the point o, with the local equation again decomposed into homogeneous coordinates as

$$f = q_\mu + q_{\mu+1} + \ldots + q_m,$$

then we ask that $(z_0 : \ldots : z_{M+1})$ forms a set of affine coordinates on $E_{\mathbb{P}}$ and the hypersurface E is given by the equation $q_\mu|_{E_{\mathbb{P}}} = 0$. This condition is to ensure that at singular points on the cover, we can apply the generalised $4n^2$-inequality, Theorem 2.3.9.

Now let $\underline{f} = (f, g) \in \mathcal{F}$ be a defining pair for a Fano cyclic cover F, $p \in F$ an arbitrary point, and let $p' = \sigma(p)$. We choose a system of affine coordinates $z_1, \ldots, z_{M+1}$ with the origin at the point p'. Without loss of generality we can assume that $z_1 = x_i/x_0$. We set $y = u/x_0^l$.

Then the standard affine set

$$\mathbb{A} = \mathbb{A}^{M+2}_{(z_1,\ldots,z_{M+1},y)}$$

is a chart for $\mathbb{P}(1, \ldots, 1, l)$. Abusing notation, we use the same symbols corresponding to the homogeneous polynomials f and g, namely $f = q_1' + q_2' + \ldots + q_m'$ at non-singular points of the intersection, $f = q_\mu + q_{\mu+1} + \ldots + q_m$ at singular points, and $g = w_0 + w_1 + \ldots + w_{Kl}$, where the polynomials q_i', q_j and w_k are homogeneous components of degree i, j and k respectively in the variables z_*, so that in the affine chart $\mathbb{A}$, the variety F is given by the pair of equations $f = 0$, $y^K = g$, replacing our original system. If the point $p \in F$ does not lie on the ramification divisor, then we assume that $w_0 = 1$. If it does, the point $p' \in G$ is non-singular, and so without loss of generality, we assume that $q_1 \equiv z_{M+1}$.

We now formulate the regularity condition for any point $o \in F$.

(R1.1) **The regularity condition for a point p outside the ramification divisor.** We begin by giving the regularity condition for a singular point. Let the singularities of F be given by the set

$$\operatorname{Sing} F = \{o_1, o_2, \ldots, o_K\}$$

where the points $o_1, \ldots, o_K$ are the K points in the preimage of the singular point $o \in G$. Let p be one of the points $o_i \in \operatorname{Sing} F$. We assume locally $w_0 = 1$ and we may also assume that $y(p) = 1$. Set

$$\begin{aligned} g^{1/K} = (1 + w_1 + \ldots + w_{Kl})^{1/K} &= 1 + \sum_{i=1}^{\infty} \gamma_i (w_1 + \ldots + w_{Kl})^i \\ &= 1 + \sum_{i=1}^{\infty} \Phi_i(w_1, \ldots, w_{Kl}), \end{aligned}$$

where $\gamma_i \in \mathbb{Q}$ are the coefficients in the Taylor expansion of $(1+s)^{1/K}$ at zero and $\Phi_i(w_1(z_*), \ldots, w_{Kl}(z_*))$ are homogeneous polynomials of degree $i \geq 1$ in the variables z_*. It is easy to see that for $i \in \{1, \ldots, Kl\}$ we get

$$\Phi_i(w_*(z_*)) = \frac{1}{K} w_i + \Phi_i^\sharp(w_1, \ldots, w_{i-1}) \tag{3.2}$$

for some polynomials $\Phi_i^\sharp$ only depending on the polynomials $(w_1, \ldots, w_{i-1})$.

In these notations we formulate the regularity condition in the following way: We say a sequence *satisfies the regularity condition* (R1.1) if the set of polynomials

$$\{q_\mu, \ldots, q_m,\ \Phi_{l+1}(w_*(z_*)), \ldots, \Phi_\nu(w_*(z_*))\} \tag{3.3}$$

forms a regular sequence in $\mathcal{O}_{p,\mathbb{C}^{M+1}}$, where

$$\nu = \begin{cases} \frac{Kl}{2} + 1 & \text{if } Kl \text{ is even} \\ \frac{Kl+1}{2} & \text{if } Kl \text{ is odd.} \end{cases}$$

In other words, the set of homogeneous equations

$$\{q_i = 0,\ \Phi_j = 0 \,|\, i = \mu, \ldots, m,\ j = l+1, \ldots, \nu\}$$

defines a closed set of codimension $\nu + m - \mu - l + 1$ in $\mathbb{C}^{M+1}$.

Considering now a non-singular point $p \neq o_i$, $i \in \{1, \ldots, K\}$ lying off the ramification divisor, the conditions are identical to those in the paper [55]. Let $u_1, \ldots, u_{M+1}$ be a system of affine coordinates with the origin at p. We perform the same decomposition as before, finishing with local equations

$$\{q'_1, \ldots, q'_m,\ \Phi'_1(w'_*(u_*)), \Phi'_2(w'_*(u_*)), \ldots\}$$

at the point p. We then require the regularity of the sequence

$$\{q'_1, \ldots, q'_m,\ \Phi'_{l+1}(w'_*(u_*)), \ldots, \Phi'_{Kl-1}(w'_*(u_*))\} \tag{3.4}$$

if $m \leq Kl$, whilst if $m > Kl$, we require the regularity of the sequence

$$\{q'_1, \ldots, q'_{m-1},\ \Phi'_{l+1}(w'_*(u_*)), \ldots, \Phi'_{Kl}(w'_*(u_*))\}. \tag{3.5}$$

Note that we also call this condition (R1.1).

(R1.2) **The regularity condition for a point p on the ramification divisor.**

Here $w_0 = 0$. We require that the set of polynomials

$$\{q_1'', \dots, q_m'', w_1'', \dots, w_K''\} \tag{3.6}$$

forms a regular sequence in $\mathcal{O}_{p,\mathbb{C}^{M+1}}$, where q_i'' and w_i'' are the local defining equations at the point p.

Definition 3.2.1. — A Fano cyclic cover defined by $\underline{f} \in \mathcal{F}$ is said to be *regular*, if every point p in the corresponding variety F satisfies the regularity conditions, namely the conditions (R1.1) and (R1.2), and the singularity on the base satisfies the condition (R0.1).

We denote the set of regular cyclic covers by the symbol $\mathcal{F}_{\mathrm{reg}}$. This is clearly open in $\mathcal{F}$.

Theorem 3.2.2. — *The set $\mathcal{F}_{reg}$ is non-empty and the following inequality holds:*

$$\operatorname{codim}(\mathcal{F} \setminus \mathcal{F}_{\mathrm{reg}} \subset \mathcal{F}) \geq 2.$$

Remark 3.2.3. — In general, when we are using the method of hypertangent divisors, we would prefer to be able to restrict to (multi)quadratic singularities, at which point we have enough "room" to be able to get an "effective" bound on the codimension where the parameter space fails to be regular. One such paper where these ideas were shown to fruition is [23], where a double covers of hypersurfaces were shown to be effectively rigid - i.e. the equivalent statement about the bound of the codimension is quadratic in the dimension of the variety F. Unfortunately, for a general cyclic cover this implies the degree of the ramification divisor to be equal to one, and so is of little use to us.

We postpone the proof of Theorem 3.2.2 for now, and first of all show the proof of 3.1.2.

3.3. Proof of 3.1.2

Assuming Theorem 3.2.2, we can now prove Theorem 3.1.2; we show that any regular cyclic cover F corresponding to a point $\underline{f} \in \mathcal{F}_{\mathrm{reg}}$ is birationally superrigid.

3.3.1. The maximal singularity. — We fix a pair of polynomials $\underline{f} = (f, g) \in \mathcal{F}_{\mathrm{reg}}$. Let $F = \mathbb{V}(f, g)$ be the corresponding cyclic cover. We recall that by our choices of m, μ, K and l, we have that

$$\operatorname{Pic} F = \mathbb{Z}H, \quad K_F = -H,$$

where H is the σ-pullback of a hyperplane section of G. Assume that F is not birationally superrigid. By what was said before, this implies that on F there is a mobile linear system $\Sigma \subset |nH|$, $n \geq 1$, with a maximal singularity E: for some non-singular projective variety $\tilde{F}$ with a birational morphism $\phi : \tilde{F} \to F$ there exists a ϕ-exceptional prime divisor $E \subset \tilde{F}$ satisfying the Noether-Fano inequality

$$\operatorname{ord}_E \Sigma > na(E).$$

Let $B = \phi(E) \subset F$ be the centre of the divisor E on F. This is an irreducible subvariety satisfying the inequality $\operatorname{mult}_B \Sigma > n$. By the corollary to the Lefschetz theorem for numerical Chow groups of algebraic cycles on V, Corollary 1.3.12, we have the equality $A^2 V = \mathbb{Z}H^2$. We can now exclude the simplest case where $\operatorname{codim}(B \subset F) = 2$.

If $\operatorname{codim}(B \subset F) = 2$, then we begin by intersecting with a general six dimensional linear subvariety $V \subset \mathbb{P}^*$, so that we have a five dimensional non-singular variety $F_V = F \cap V$. Further set $H_V = H \cap V$. Then by the Lefschetz theorem,

$$\operatorname{Pic} F_V = \mathbb{Z}H_V, \quad A^2 F_V = \mathbb{Z}H_V^2.$$

If we then also set $B_V = B \cap V$, then $B_V \sim mH_V^2$ for some $m \geq 1$. Consider the self intersection $Z = (D_1 \circ D_2)$ of the linear system $\Sigma_V = \Sigma \cap V$. Clearly $Z \sim n^2 H_V^2$.

On the other hand, $Z = \gamma B_V + Z_1$ where $\gamma > n^2$ and Z_1 is an effective cycle of codimension 2 that does not contain B_V as a component (here we are using the Noether-Fano inequality together with, for example, [59, Chapter 2, Lemma 2.2]). Taking the classes of the cycles in A^2F_V then yields the inequality $n^2 \geq \gamma m > mn^2$. This is the required contradiction.

If $\operatorname{codim}(B \subset F) \geq 3$ and $B \not\subset \operatorname{Sing} F$, then the inequality $\operatorname{mult}_B Z > 4n^2$ holds. This is the classical $4n^2$-inequality going back to [35] (see [59, Chapter 2] for a modern exposition). At this point, we use the arguments of [59, Chapter 3, Section 2, Theorem 2.1] to cover this case. We can do this because it relies only on the regularity conditions at non-singular points.

We are therefore left with the only option: B is a singular point lying off the ramification divisor, specifically $B = p \in \{o_1, \ldots, o_K\}$. To exclude the remaining case, we use the method of hypertangent linear systems.

3.3.2. Hypertangent linear systems — We are now in the position to make use of the technique of hypertangent linear systems.

Returning to our original cover

$$\sigma : F \to G,$$

set

$$D_i = \sigma^* \overline{\{(q_\mu + \ldots + q_i)|_G = 0\}}$$

where $i = \mu, \ldots, m-1$ and where we are taking the closure in $\mathbb{P}^*$. Similarly, let

$$L_j = \overline{\left\{ y - 1 - \sum_{i=1}^{j} \Phi_i(w_1, \ldots, w_j)|_F = 0 \right\}}$$

where $j = l, \ldots, Kl - 1$. These sets are clearly both of hypertangent divisors with multiplicities at the point p of:

$$\operatorname{mult}_p D_i = i + 1, \quad \operatorname{mult}_p L_j = j + 1,$$

and hence with slopes

$$\beta(D_i) = \frac{i+1}{i}, \quad \beta(L_j) = \frac{j+1}{j}$$

respectively. To see this note that $(q_\mu + \ldots + q_i)|_G = (-q_{i+1} - \ldots - q_m)|_G$ and similarly for the divisors L_j. Define the set

$$\mathcal{D} = \{D_i \,|\, i = \mu, \ldots, m-1\} \cup \{L_j \,|\, j = l \ \ldots, \nu - 1\}.$$

to be the collection of these hypertangent divisors. Let

$$N = \#\mathcal{D} = m - \mu + \nu - l.$$

We now generate a hypertangent linear system in the usual way: Let

$$\pi : X^+ \to X$$

be the blow up of the variety X at the point p where E^+ denotes the strict transform of an arbitrary divisor E. Then for each $D \in \mathcal{D}$, define k_D and l_D to be the coefficients in the expression of the strict transform $D^+ \in |k_D H - l_D E|$. Let

$$f_D \in H^0(X, \mathcal{O}_X(k_D H))$$

be a section defining the divisor D. Set

$$\Sigma_k = \left| \sum_{k_D \leq k} f_D s_D = 0 \right|$$

where the summation is taken over all hypertangent divisors $D \in \mathcal{D}$ such that $k_D \leq k$, and s_D is the pullback of an arbitrary polynomial of degree $(k - k_D)$ with a zero of order $(k - k_D)$ at the point $\alpha(p)$. Clearly

$$\beta(\Sigma_k) \geq \min_{D \in \mathcal{D},\, k_d \leq k} \left\{ \frac{k + l_D - k_D}{k} \right\}.$$

In fact, in our case, for every $D \in \mathcal{D}$ we have that $l_D = k_D + 1$, so that, as expected all the slopes are of the form

$$\beta(\Sigma_k) \geq \frac{k+1}{k}.$$

From this equality, we see that the integer-valued function $\operatorname{codim}_o \operatorname{Bs}\Sigma_k$ is increasing when $k = k_D$ for some $D \in \mathcal{D}$, and only for those values.

As before we have the *ordering function*

$$\chi : \{1, \ldots, N\} \to \mathcal{K} = \{k_D \mid D \in \mathcal{D}\}$$

defined by the relation

$$\#\{D \in \mathcal{D} \mid k_D < \chi(i)\} < i \leq \#\{D \in \mathcal{D} \mid k_D \leq \chi(i)\}. \tag{3.7}$$

By construction, we again obtain

$$\operatorname{codim}_o \operatorname{Bs}\Sigma_{\chi(i)} \geq i.$$

From this, pick a general set of hypertangent divisors

$$\mathbb{D} = (D_1, \ldots, D_N) \in \prod_{i=1}^{N} \Sigma_{\chi(i)}$$

and an arbitrary subvariety Y of codimension d containing the point p; we get $Y \not\subset \operatorname{Supp}(D_i)$ for $i \geq d+1$. In particular, let us take Y to be an irreducible component of the self-intersection Z with the highest ratio $\operatorname{mult}_p / \deg$ of the multiplicity at the point p to the degree d; this is clearly bounded above by 1. We can now construct in the usual way (see [59, Chapter 3]) a sequence of irreducible subvarieties $Y_2 = Y$, an arbitrary codimension 2 subvariety of X, $Y_3, \ldots, Y_N$ satisfying the following properties:

- $\operatorname{codim}(Y_i \subset F) = i$,
- $Y_i \not\subset D_{\chi(i+1)}$, so that $(Y_i \circ D_{\chi(i+1)})$ is an effective cycle on V and Y_{i+1} is one of

its irreducible components,

- Y_{i+1} is an irreducible component of the algebraic cycle of the scheme-theoretic intersection $(Y_i \circ D_{\chi(i+1)})$ with the maximal possible value of $\operatorname{mult}_o Y_{i+1} / \deg Y_{i+1}$.

In particular, the inequality

$$\frac{\operatorname{mult}_p Y_{i+1}}{\deg Y_{i+1}} \geq \beta(\Sigma_{i+1}) \frac{\operatorname{mult}_p Y_i}{\deg Y_i}$$

holds, again using Lemma 1.3.16, which immediately implies the following proposition:

Proposition 3.3.3. — *The following inequality holds:*

$$\frac{\operatorname{mult}_p Y}{\deg Y} \leq \left(\prod_{i=1}^{N-2} \beta(\Sigma_{i+2}) \right)^{-1}$$

This implies the following inequality:

$$\frac{\operatorname{mult}_o Y}{\deg Y} \leq \frac{4\mu}{mK}, \tag{3.8}$$

which holds at every singular point $o \in \operatorname{Sing}(F)$ including p.

Theorem 3.1.2 then follows if this holds true; to see this, note that F being birationally superrigid would imply that $\deg Z = mKn^2$ and $\operatorname{mult}_p Z > 4\mu n^2$, contradicting the above.

If we now apply Proposition 3.3.3 to the subvariety Y, we can hence deduce the following:

$$\begin{aligned} \frac{\operatorname{mult}_p Y}{\deg Y} &\leq \left(\frac{\mu}{\mu+1} \cdot \frac{\mu+1}{\mu+2} \cdot \prod_{i=\mu}^{m-1} \frac{i+1}{i} \cdot \prod_{j=l}^{\nu-1} \frac{j+1}{j} \right)^{-1} \\ &= \frac{\mu+2}{\mu} \cdot \frac{\mu \cdot l}{m\nu} < \frac{(\mu+2) \cdot 2}{mK} \leq \frac{4\mu}{mK}. \end{aligned}$$

If F were birationally superrigid with a centre of a maximal singularity at the singular point p, however, this would imply the existence of a codimension 2 subvariety Z, specifically the self-intersection of a mobile linear system $\Sigma \subset |nH|$, with the following properties: $\deg Z = mKn^2$ and $\operatorname{mult}_p Z > 4\mu n^2$. However, this clearly contradicts the above, so we have been able to exclude the possibility that the singular point p is a singularity, and hence, since we have previously exhausted all the other cases, the theorem follows accordingly.

3.4. Proof of Theorem 3.2.2.

Let us now prove Theorem 3.2.2. This follows from the following: $\mathcal{F}$ is non-empty - we show this in the proof of 3.4.6. Now let $p \in \mathbb{P}^*$ be an arbitrary fixed point (not necessarily the same as the point p from the previous section), and let $p' = \sigma(p)$. Consider the set $\mathcal{F}(p) = \{\underline{f} \in \mathcal{F} \mid G \ni p'\} \subset \mathcal{F}$. Since the cover σ is cyclic, either all the points $\sigma^{-1}(p')$ satisfy the regularity conditions, or none of them do. Set

$$\mathcal{F}_{\mathrm{reg}}(p) \subset \mathcal{F}(p)$$

to be the set of covers such that each point $p' \in \sigma^{-1}(p)$ satisfies the regularity conditions.

Proposition 3.4.1. — *The following inequality holds:*

$$\operatorname{codim}_{\mathcal{F}(p)}(\mathcal{F}(p) \backslash \mathcal{F}_{\mathrm{reg}}(p) \subset \mathcal{F}(p)) \geq M + 2.$$

Since $p \in \mathbb{P}^{M+1}$ is an arbitrary point, and $\mathcal{F}(p) \subset \mathcal{F}$ is a divisor, we can use the same argument as in the case of a Fano complete intersection (see [59, Chapter 3, Section 3, 3.2]) to complete the proof of Theorem 3.2.2.

First of all, the case where p is non-singular has been covered in [55, Proposition 5.1]. Therefore, we assume that p is singular (and hence has multiplicity μ).

We need the following lemma. We construct a sequence of polynomials $\Phi_i^+(w_1, \ldots, w_l)$, first of all setting

$$\Phi^+_{l+1} = \Phi^\sharp_{l+1}$$

where the polynomials $\Phi^\sharp_j$ were defined as in the equation 3.2. We then subsequently set

$$\Phi^+_i = \Phi^\sharp_i(w_1, \dots, w_l, -K\Phi^+_{l+1}, \dots, -K\Phi^+_{i-1})$$

for $i \geq l+2$.

Lemma 3.4.2. — *The sequence 3.3 is regular in the ring $\mathcal{O}_{F,p}$ if and only if the set of polynomials*

$$\{q_\mu, \dots, q_m,\ w_{l+1} + K\Phi^+_{l+1}, \dots, w_\nu + K\Phi^+_\nu\} \tag{3.9}$$

forms a regular sequence.

Proof. The sets of zeros for both sequences are the same: this can be shown by induction using the equality

$$\Phi_i|_B \equiv 0 \iff w_i|_B \equiv -K\Phi^\sharp_i$$

for an arbitrary closed irreducible set B. □

Note that the sequence 3.9 has the polynomials $w_i(z_*)$ shifted by polynomials Φ^+_i which depend only on $w_1, \dots, w_l$. The set of polynomials w_i, $i \in \{1, \dots, l\}$ can be assumed to be fixed, and all taken to be general. Therefore in the sequence 3.9 each of the homogeneous polynomials w_i, $i \in \{l+1, \nu\}$ is shifted by a fixed homogeneous polynomial of degree i.

Let Π be the space of polynomials $q_\mu, \dots, q_m$, $w_1, \dots, w_\nu$. Consider an irreducible component $X \subset \Pi$ corresponding to non-regular sequences 3.3 or 3.9. For a fixed set of homogeneous polynomials $u_1, \dots, u_l$, where $\deg u_i = i$, let

$$\Pi(u_1, \dots, u_l) = \{w_i = u_i |\, i = 1, \dots, l\} \subset \Pi$$

be the corresponding affine subspace with fixed polynomials $w_1, \ldots, w_l$, and set

$$X(u_1, \ldots, u_l) = X \cap \Pi(u_1, \ldots, u_l).$$

Note that $\Pi(u_1, \ldots, u_l)$ is identified with the space of polynomials $q_\mu, \ldots, q_m$, $w_{l+1}, \ldots, w_\nu$, which we denote by Π^+. Thus we consider $X(u_1, \ldots, u_l)$ to be embedded in the linear space Π^+. For a general tuple $(u_1, \ldots, u_l)$ we have

$$\operatorname{codim}_\Pi X = \operatorname{codim}_{\Pi^+} X(u_1, \ldots, u_l).$$

Lemma 3.4.3. — $X(0, \ldots, 0) \neq \emptyset$.

Proof. Φ_i, $\Phi_i^\sharp$ and Φ_i^+ are quasi-homogeneous in w_*, where the coordinates w_* are weighted so that $\operatorname{wt} w_i = i$ for every value of i. Therefore, for $\lambda \neq 0$,

$$(q_\mu, \ldots, q_m, \lambda^{l+1} w_{l+1}, \ldots, \lambda^\nu w_\nu) \in X(\lambda u_1, \lambda^2 u_2, \ldots, \lambda^l u_l)$$

if and only if

$$(q_\mu, \ldots, q_m, w_{l+1}, \ldots, w_\nu) \in X(u_1, \ldots, u_l).$$

Setting $\lambda = 0$ gives us the statement, using closure of the component. $\square$

Remark 3.4.4. — If we then notice that

$$\operatorname{codim}_\Pi X \geq \operatorname{codim}_{\Pi^+} X(0, \ldots, 0),$$

this allows us to calculate the codimension of the space where the regularity conditions fail, essentially by ignoring the polynomials Φ_i^+, and so we only need to estimate the codimension of the closed set of non-regular sequences $q_\mu, \ldots, q_m$, $w_{l+1}, \ldots, w_\nu$.

We should however take into account the effect that introducing the singularity will affect the proof of regularity of non-singular points in the same neighbourhood. In fact, we consider the larger set of non-regular sequences $q_\mu, \ldots, q_m$, $w_{l+1}, \ldots, w_{Kl}$, as the bound is good enough for our purposes.

3.4.5. Codimension estimate — Begin first of all by noting that the codimension estimate is trivial at the point o, and follows from the usual argument for a codimension count in the non-singular case. However, as we will see, it is possible the singular point will have an effect on the codimension of the set of non-regular sequences at nearby non-singular points, so we have to check that the regularity conditions hold here as well. Recall that we are working in the chart $\mathbb{A}$ where the y coordinate is fixed to be equal to 1, and by abuse of notation whenever we talk about the variety F, we are referring to its restriction to the chart $\mathbb{A}|_{\{y=1\}}$. We let $\mathcal{P}_{d,M+1}$ stand for the linear space of homogeneous polynomials of degree d in $M+1$ variables (z_*). Set

$$\mathcal{P}_{[a,b],\,M+1} = \prod_{l=a}^{b} \mathcal{P}_{l,\,M+1}$$

to be the space of tuples of polynomials of the form $(q_a, q_{a+1}, \ldots, q_b)$, where $q_d \in \mathcal{P}_{d,M+1}$. We then let

$$\mathcal{P} = \mathcal{P}_{[\mu,m],\,M+1} \times \mathcal{P}_{[l+1,Kl],\,M+1}$$

to be the space of pairs $\underline{f}$ of defining polynomials of the type discussed above. In the following, F always refers to the corresponding variety. Note that for a general pair the condition (R0.1) is satisfied.

Let $o \in \mathrm{Sing}(F)$ be a singular point lying at the origin in $\mathbb{A}$ and let $p \in \mathbb{A}$, $p \notin \mathrm{Sing}(F)$ be an arbitrary point. We assume that p has coordinates $(1, 0, \ldots, 0)$. We let

$$u_* = \{u_1 = z_1 - 1,\, u_2 = z_2, \ldots, u_{M+1} = z_{M+1}\}$$

be a system of affine coordinates with origin at the point p. Set $q_j = q_{j,k} + z_1 q_{j,k-1} + \ldots + z_1^k q_{j,0}$ and $w_j = w_{j,k} + z_1 w_{j,k-1} + \ldots + z_1^k w_{j,0}$ where $q_{j,k}$ and $w_{j,k}$ are homogeneous polynomials of degree $k \leq j$ in the variables $z_2, \ldots, z_{M+1}$. In the alternate coordinates u_*, we can see that the polynomial q'_j takes the form

$$\sum_{\alpha=0}^{j} u_1^{j-\alpha} \left(\sum_{\delta=\max(\mu,j)}^{m} \binom{\delta-\alpha}{j-\alpha} q_{\delta,\alpha} \right). \tag{3.10}$$

Similarly, w'_j takes the form

$$\sum_{\alpha=0}^{j} u_1^{j-\alpha}\left(\sum_{\delta=j}^{Kl}\binom{\delta-\alpha}{j-\alpha}w_{\delta,\alpha}\right). \tag{3.11}$$

Therefore, the change of coordinates $(z_*) \to (u_*)$ defines a linear map

$$\tau : \mathcal{P} \to \mathcal{P}_{[0,m],\,M+1}(u_*) \times \mathcal{P}_{[0,Kl],\,M+1}(u_*).$$

Theorem 3.4.6. — *The set of pairs $\underline{f}$ in $\mathcal{P}$ such that*

$$\tau(\underline{f}) \in \mathcal{P}_{[1,m],\,M+1}(u_*) \times \mathcal{P}_{[0,Kl],\,M+1}(u_*)$$

where the second polynomial has constant term equal to 1 *and* $\tau(\underline{f})$ *fails the regularity condition (R1.1) is of codimension at least* $M+2$ *in the space* $\mathcal{P}$.

Proof. We begin by noting that $q_0 = q_{\mu,0} + \ldots + q_{m,0}$, $w'_0 = w_{0,0} + \ldots + w_{Kl,0}$ and the equalities $q'_0 = 0$, $w'_0 = 1$, expressing the fact that p lies on the vanishing set of f and $g-1$, give 2 independent conditions for the polynomials f and g; we then have remaining $M+1+l$ degrees of freedom for the non-linear defining polynomials q'_i and w'_i. This immediately gives us non-emptiness of the set $\mathcal{F}$.
For a fixed linear form L in the variables u_* we denote by the symbol

$$\mathcal{P}_{[p;\,L]} \subset \mathcal{P}$$

the affine subspace of pairs $\underline{f}$ such that $q'_0 = 0$, $w'_0 = 1$ and $q'_1 = L$.

3.4.7. The line connecting the points o and p. — Let us denote this line by the symbol $[o,p]$. We say we are in the *non-special case* if $[o,p] \not\subset T_pF$, and in the *special* case if $[o,p] \subset T_pF$. First of all we note that in the coordinates u_*, the line is given by the equations

$$u_2 = u_3 = \ldots = u_{M+1} = 0$$

on the space $T_p\mathbb{P}^*$. Supposing $[o,p] \not\subset T_pF$, we then have

$$\dim\langle u_2|_{T_pF}, \ldots, u_{M+1}|_{T_pF}\rangle = M - 1$$

so that for every $j \geq 1$ the space

$$\{q_j|_{T_pF} \,|\, q_j \in \mathcal{P}_{j,M}\}$$

(where $\mathcal{P}_{j,M}$ uses the M variables not equal to z_1) is the whole space of homogeneous polynomials of degree j on T_pF. Note that it follows from the equation 3.10 that $q'_\delta = q_{m,\delta} + (*)$, where $(*)$ is a linear combination of terms $u_1^{\delta-k}q_{j,k}$ where either $j < m$ or $j = m$ and $k < \delta$.

As the polynomials $q_{j,k}$ are arbitrary of degree k in the variables $u_2, \ldots, u_{M+1}$, we conclude that when we consider the pairs $\underline{f}$ in the space $\mathcal{P}_{[p;\,L]}$, there is no dependence between any of the polynomials $q'_\delta|_{T_pV}$ (and indeed the polynomials $w'_\delta|_{T_pV}$) where $\delta \geq 2$. This means that the methods outlined in [59, Chapter 3] and so we can use the result on non-singular cyclic covers from [55, Proposition 5.1] (noting that the slight difference in regularity conditions - we go up to w_{Kl} instead of w_{Kl-1} - has no bearing on the final estimate) to get the desired inequality, that the set of pairs $\underline{f}$ in $\mathcal{P}$ which fail the regularity condition (R1.1) has codimension at least $M+1$ in the space $\mathcal{P}[p;L]$. Since for different linear forms $L \neq L'$ the spaces $\mathcal{P}_{[p;\,L]}$ and $\mathcal{P}_{[p;\,L']}$ are disjoint, we essentially reduce to the case where the cyclic cover is non-singular, and where the codimension estimate is as required, again using [55, Proposition 5.1].

Therefore, we assume we are in the special case, that is, $L|_{[o,p]} \equiv 0$. Explicitly, this means that the equality

$$\mu q_{\mu,0} + \ldots + m q_{m,0} = 0$$

holds, so we can no longer directly use the previous result. Now note that if $\mu \leq m-1$ we obtain a new independent condition on $\underline{f}$. We then use the following proposition: We set $\mathbb{T} = \mathbb{P}(T_pF) \cong \mathbb{P}^{M-1}$. Within this space, we denote the point corresponding to the line $[o, p]$ by ω.

Proposition 3.4.8. — *Suppose we are in the special case. Then the set of pairs* $\underline{f} \in \mathbb{P}$ *such that the system of equations*

$$q'_j|_{\mathbb{T}} = 0, \quad w'_k|_{\mathbb{T}} = 0,\, 2 \leq j \leq m,\, l \leq k \leq Kl - 2$$

has in $\mathbb{T}$ *a positive-dimensional set of solutions, is of codimension at least*

$$\begin{cases} M+1 & \textit{if } \mu = m \\ M & \textit{if } \mu \leq m-1 \end{cases}$$

in the space $\mathcal{P}[p; L]$.

Proof. Let us begin by fixing the linear space L, and hence the projective space $\mathbb{T}$. Placing the polynomials $q'|_{\mathbb{T}}$ and $w'|_{\mathbb{T}}$ in lexicographic order we get $M-1$ polynomials on $\mathbb{P}^{M-1}$:

$$p_1, p_2, \ldots, p_{M-1},$$

where $\deg p_{i+1} \geq \deg p_i$. As we are in the special case it is no longer true that the p_i run through the corresponding spaces of polynomials independently of each other, so we can no longer use the standard method, outlined in [59, Chapter 3, Section 3].

Therefore let us consider the affine space $\mathcal{A} = \mathcal{P}[p; L]$. Let $\mathcal{B}_{\text{line}} \subset \mathcal{A}$ be the set of pairs $\underline{f} \in \mathcal{A}$ such that $p_i|R \equiv 0$ for some line $R \subset \mathbb{T}$ for every i. Furthermore, set $\mathcal{B}_i \subset \mathcal{A} \setminus \mathcal{B}_{\text{line}}$ to be the set of pairs $\underline{f} \in \mathcal{A}$ such that

$$\operatorname{codim}(\{p_1 = \ldots = p_{i-1} = 0 \subset \mathbb{T}\}) = i - 1,$$

but for some irreducible component B of the set $\{p_1 = \ldots = p_{i-1} = 0\}$ we have $p_i|_B \equiv 0$. (For $i = 1$ this condition means that $p_i \equiv 0$).

We then need the following pair of propositions:

Proposition 3.4.9. — *The following inequality holds:*

$$\operatorname{codim}(\mathcal{B}_{line} \subset \mathcal{A}) \geq M + c_* - 1,$$

where

$$c_* = \begin{cases} 1 & \text{if } m = \mu \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.4.10. — *For all* $i = 1, \ldots, M-1$ *the following inequality holds:*

$$\operatorname{codim}(B_i \subset \mathcal{A}) \geq M + 1.$$

The estimate of the codimension then clearly follows from these.

Remark 3.4.11. — Let $(v_*) = (v_0 : v_1 : \ldots : v_{M-1})$ be a system of homogeneous coordinates on $\mathbb{T}$, and let ω be given by the point $(1 : 0 : \ldots : 0)$. The formulas 3.10 and 3.11 imply that for fixed polynomials $p_1, \ldots, p_{i-1}$ the set through which the polynomial p_i runs is a disjoint union of affine subspaces of the form

$$p_i' + \mathcal{P}_{\deg p_i, M-1}(v_1, \ldots, v_{M-1}),$$

where p_i' is some polynomial. Applying the method of linear projections from [59, Chapter 3, Section 3], we obtain the inequality

$$\operatorname{codim}(\mathcal{B}_i \subset \mathcal{A}) \geq \binom{M - i - 1 + \deg p_i}{\deg p_i}.$$

Note that when $i = 1, 2$, this already gives us what we need:

$$\operatorname{codim}(\mathcal{B}_i \subset \mathcal{A}) \geq \binom{M-1}{2}$$

Therefore it is sufficient to prove Proposition 3.4.10 for $i \geq 3$, so that $\deg p_i \geq 3$.

3.4.12. Proof of Proposition 3.4.9 — We consider the subsets of B_{line} corresponding to the case when R contains the point ω, $(\mathcal{B}^+_{\text{line}})$, and when it does not

$(\mathcal{B}^{-}_{\text{line}})$. We estimate the codimensions of these sets in $\mathcal{A}$ separately.

For $\underline{f} \in (\mathcal{B}^{-}_{\text{line}})$ the conditions $p_i|_R \equiv 0$, $i = 1, \ldots, M-1$ similarly to the non-special case give $\sum_{i=1}^{M-1}(\deg p_i + 1)$ independent conditions for $\underline{f}$. Since the line R varies in an $2(M-2)$-dimensional family, we obtain the estimate

$$\operatorname{codim}(\mathcal{B}^{-}_{\text{line}} \subset \mathcal{A}) \geq \sum_{i=1}^{M-1} \deg p_i - M + 3.$$

Lemma 3.4.13. — *The following inequality holds:*

$$\sum_{i=1}^{M-1} \deg p_i \geq 2M - 2.$$

Proof. Note that

$$\sum_{i=1}^{M-1} \deg p_i = \frac{m(m+1)}{2} - 1 + \frac{(Kl-1)(Kl-2)}{2} - \frac{l(l+1)}{2} \tag{3.12}$$

Note that if $m \leq Kl - 3$ (or vice versa), then we can replace m by $m+1$ and Kl by $Kl - 1$ and this does not increase the value of either expression in the formula 3.12. Further, we can assume that we are in the worst possible case, that is where we have $l = 1$. Therefore, the minimum of the expression is obtained when the values of m and $Kl - 1$ are as close as possible, that is when

$$m = a + 1, \; Kl = a + 3$$

where $M = 2a + r$, $r \in \{0, 1\}$. □

The above lemma then implies the inequality

$$\operatorname{codim}(\mathcal{B}^{-}_{\text{line}} \subset \mathcal{A}) \geq M + 1,$$

which is stronger than the inequality needed (we only need our inequality to be greater than or equal to M). We then need to consider the other case ($\mathcal{B}^+_{\text{line}} \subset \mathcal{A}$), which follows from the following claim:

Proposition 3.4.14. — *The following inequality is true:*

$$\operatorname{codim}(\mathcal{B}^+_{line} \subset \mathcal{A}) \geq M + 1 + c_* - k.$$

Proof. Let $R \ni \omega$ be a line. In the notations of Remark 3.4.11 let

$$\lambda = (0 : a_1 : \ldots : a_{M-1}) = R \cap \{v_0 = 0\}.$$

The conditions $p_i|_R \equiv 0$, $i = 1, \ldots, M-1$ give a smaller codimension that in the case $R \not\ni \omega$ considered above. However, on the other hand the lines R containing ω vary in a $(M-2)$-dimensional family. Let us fix the line R and the point λ.
At this point we suppose that $m \leq Kl$ (the opposite case is almost identical, swapping q'_* for w'_* where appropriate).

Lemma 3.4.15. — *The conditions*

$$q'_2|_R \equiv \ldots \equiv q'_m \equiv 0$$

are equivalent to the conditions

$$q_{j,k}(\lambda) = 0.$$

where $\mu \leq j \leq m$, $k = 0, 1 \ldots, j$.

Proof. For the homogeneous polynomial

$$q'(v_*) = v_0^l r_0 + v_0^{l-1} r_1 + \ldots + v_0 r_{l-1} + r_l$$

where $r_i(v_1, \ldots, v_{M-1})$ is a homogeneous polynomial of degree i, the condition $q'|_R \equiv 0$ means that

$$r_0 = r_1(\lambda) = \ldots = r_l(\lambda) = 0.$$

The formula 3.10 implies that if all polynomials q'_j vanish identically on the line R, then the equalities

$$\sum_{j=\max(\mu,e)}^{m} \binom{j-k}{e-k} q_{j,k}(\lambda) = 0$$

hold for every $e = 0, \ldots, m$ and $k = 0, \ldots, e$. Setting $e = m$, we obtain the system of equalities

$$q_{m,k}(\lambda) = 0,\ k = 0, \ldots, m.$$

If $\mu = m$, then the claim is shown.
Assume instead that $\mu \leq m-1$. Setting $e = m-1$, we obtain the system of equalities

$$q_{m-1,k}(\lambda) + \binom{m-k}{m-1-k} q_{m,k}(\lambda) = 0$$

for $k = 0, \ldots, m-1$, whence, taking into account the previous inequalities, we conclude that

$$q_{m-1,k}(\lambda) = 0,\ k = 0, \ldots, m-1.$$

Similarly, we do the same for the values $k = m-2, \ldots, \mu$ and complete the proof of the lemma. $\square$

Lemma 3.4.16. — *The conditions*

$$w'_2|_R \equiv \ldots \equiv w'_{Kl-2}|_R \equiv 0$$

define a linear subspace of codimension

$$\frac{1}{2}[(Kl-1)(Kl-2)] - 1$$

in the space of tuples of homogeneous polynomials $w_{j,k}$, $1 \leq j \leq Kl-2$, $k = 0, 1, \ldots, j$.

Proof. Adding the condition

$$w'_{Kl-1}|_R \equiv 0$$

and applying the previous lemma, we obtain $\frac{1}{2}[(Kl)(Kl+1)] - 1$ independent linear conditions $w_{j,k}(\lambda) = 0$. The vanishing of w_{Kl-1} on the line R adds Kl linear conditions.

$\square$

Combining Lemmas 3.4.15 and 3.4.16, we see Proposition 3.4.14 follows from the inequality

$$\frac{1}{2}[(m+1)(m+2)-\mu(\mu+1)+(Kl-1)(Kl-2)-2]-(M-2)\geq M+c_*-1.$$

But this is just Equation 3.1. Thus we have proved Propositions 3.4.9 and 3.4.14. $\square$

3.4.17. Proof of Proposition 3.4.10 — Note that by Remark 3.4.11 we can assume that $\deg p_i \geq 3$. From this point we can use (a modified version of) the method of good sequences and associated subvarieties, which is described in [59, Chapter 3, Section 3].

Let $\mathcal{B}_{i,b} \subset \mathcal{B}_i$ be the subset of pairs $\underline{f} \in \mathcal{A}$ such that for some irreducible component B of the set $\{p_1 = \ldots = p_{i-1} = 0\}$ (which has codimension $i-1$ in $\mathbb{T}$, since $\underline{f} \in \mathcal{B}_i$), such that $\mathrm{codim}(\langle B\rangle \subset \mathbb{T}) = b$, we have $p_i|_B \equiv 0$. The parameter b runs through the set of values $\{0, 1, \ldots, i-1\}$ for $i \leq M-2$, and through the set $\{0, \ldots, M-3\}$ for $i = M-1$. When $b = i-1$, the component B is a linear subspace in $\mathbb{T}$, and the codimension $\mathrm{codim}(\mathcal{B}_{i,i-1} \subset \mathcal{A})$ can be calculated explicitly, though this is stronger than we need.

Let P be a linear subspace of codimension b in $\mathbb{T}$. By the symbol $\mathcal{B}_{i,b}(P)$ we denote the subset of pairs $\underline{f} \in \mathcal{B}_{i,b}$ such that the linear span of the associated irreducible subvariety B is P. Obviously

$$\mathrm{codim}(\mathcal{B}_{i,b} \subset \mathcal{A}) \geq \mathrm{codim}(\mathcal{B}_{i,b}(P) \subset \mathcal{A}) - b(M-b).$$

Furthermore, for a subset of indices

$$I = \{j_1 < \ldots, < j_{i-1-b}\} \subset \{1, \ldots, i-1\}$$

let $\mathcal{B}_{i,b,I}(P) \subset \mathcal{B}_{i,b}(P)$ be the subset of pairs $\underline{f} \in \mathcal{B}_{i,b}(P)$ such that there exists a sequence of irreducible subvarieties

$$Y_0 = P, Y_1, \ldots, Y_{i-1-b} = B$$

satisfying the following properties:

- for every $l \in \{1, \ldots, i-1-b\}$ and every index $j_{l-1} < j < j_l$ (where $j_0 = 0$) the polynomial p_j vanishes identically on Y_{l-1},
- for every $l \in \{1, \ldots, i-1-b\}$ we have $p_{j_l}|_{Y_{l-1}} \not\equiv 0$ and $Y_l \subset Y_{l-1}$ is an irreducible component of the closed set $\{p_{j_l}|_{Y_{l-1}} = 0\}$ containing the subvariety B.

In the terminology of [59] we say the polynomials $p_{j_l}|_P$, $l = 1, \ldots, i-1-b$ form a *good sequence* with B as one of its associated subvarieties. Obviously,

$$\mathcal{B}_{i,b}(P) = \bigcup_I \mathcal{B}_{i,b,I}(P).$$

Lemma 3.4.18. — *The following inequality holds:*

$$\operatorname{codim}(\mathcal{B}_{i,b,I}(P) \subset \mathcal{A}) \geq (2b+3)(M-1-b) - 2$$

Proof. We check the polynomials p_j not included in the good sequence individually. When the polynomials p_γ with $\gamma < j$ are fixed, the condition $p_j|_{Y_{l-1}} \equiv 0$ imposes on the coefficients of the polynomial p_j at least

$$\deg p_j(M-2-b) + 1 \geq 2(M-2-b) + 1$$

independent conditions, since $\langle Y_{l-1} \rangle = P$ (recall that $Y_{l-1} \supset B$). There are b of these. The condition $p_i|_B \equiv 0$ gives (with $p_1, \ldots, p_{i-1}$ fixed) at least

$$\deg p_i(M-2-b) + 1 \geq 3(M-2-b) + 1$$

independent conditions. Putting these two together completes the proof of the lemma. $\square$

We now complete the proof of Proposition 3.4.10. Let us look first of all at the values $i \leq M-2$ when the parameter b takes the values $0, 1, \ldots, i-1$. Let us consider the quadratic function

$$\phi_1(t) = (2t+3)(M-1-t) - t(M-t) - 2.$$

Since $\phi_1''(t) = -1 < 0$, its minimum on the set $[0, i-1]$ is attained either at $t = 0$ or at $t = i-1$. Therefore, for $i = k+1, \ldots, M-2$ we get

$$\operatorname{codim}(\mathcal{B}_i \subset \mathcal{A}) \geq \min\{3M-5, (M-i-1)(i+2)+1\}.$$

Since $3M-5 \geq M+1$, which is what we need, let us consider the quadratic function

$$\phi_2(t) = (M-t-1)(t+2)+1.$$

Again, $\phi_2''(t) = -\frac{1}{2} < 0$, so that its minimum on the set $[3, M-2]$ is attained at either endpoint. In the $M-2$ case we get $\phi_2(M-2) = M+1$ as required. Therefore, in order to prove Proposition 3.4.10 for $i \leq M-2$, it is sufficient to show the truth of the inequality

$$5(M-4)+1 \geq M+1.$$

But $M \geq 5$, so this follows immediately. Therefore we have proved Proposition 3.4.10 in the case $i \leq M-2$.

Finally, in the case where

$$i = M-1,$$

the parameter b takes the values $0, 1, \ldots, M-3$ (We needed Proposition 3.4.9 in order to deal with the case $b = M-2$). If $b = 0$, we get $\phi_1(0) = 3M-5$ as above which is fine. On the other hand, if $b = M-3$, we get the value $\phi_1(M-3) = M+1$, which leads to the estimate

$$\operatorname{codim}(\mathcal{B}_{M-1} \subset \mathcal{A}) \geq M+1,$$

which is also fine. This finishes off the proof of Proposition 3.4.10. and hence 3.2.2. □

Since we have now proved 3.4.10, Theorem 3.2.2 now follows.

□

4.

The Canonical Threshold of a General Cyclic Cover

In this chapter, we define the *canonical threshold* of a variety, and from this aim to find a bound for the canonical threshold of a general cyclic cover. This leads to some crossover to the case where we can consider the varieties in question in the complex analytic setting, proving the existence of a *Kähler-Einstein* metric on such a variety.

4.1. Introduction

We first take a brief detour to Complex Geometry. We need several definitions to motivate the work of this chapter. Let X be a non-singular variety, so in particular it has the structure of a complex manifold. We need the following definition.

Definition 4.1.1. — We say that X is *Kähler* if it has a Hermitian metric g such that the associated 2-form ω, where $\omega = \frac{i}{2}(h - \bar{h})$, is closed, i.e. $d\omega = 0$. The *Ricci curvature* of g is then given by

$$\mathrm{Ric} = -\frac{\partial^2}{\partial z^j \partial \bar{z}^k} \log \det(g).$$

We say that X is *Einstein* if $\mathrm{Ric} = \lambda g$ for some constant $g \in \mathbb{R}$. Finally, we say that X is *Kähler-Einstein* if it is both Kähler and Einstein.

The importance of these definitions is related to the very important property of *K-stability*, which we will not define in this thesis. In layman's terms, we can categorise when a Fano manifold is Kähler-Einstein if and only if it is also K-stable. A good introduction to these ideas can be found in [75].

We now return to the algebraic setting and define the global canonical threshold (or simply *canonical threshold*) for a variety X to be

$$\mathrm{ct}(X) = \sup\{\lambda \in \mathbb{Q}_+ | (X, \frac{\lambda}{n}D) \text{ is canonical for every } D \in |nH| \text{ for every value } n \geq 1\}.$$

Similarly, we define the global log canonical threshold by the equality

$$\mathrm{lct}(X) = \sup\{\lambda \in \mathbb{Q}_+ | (X, \frac{\lambda}{n}D) \text{ is log canonical for every } D \in |nH| \text{ for every value } n \geq 1\}.$$

The definition of the global log canonical threshold was introduced by Cheltsov and Park as [6, Definition 1.7], and was announced in a COW seminar held at Liverpool University in 2000 by Cheltsov (the author is unsure of the exact origin of the sister definition, though it is clearly very similar). It was later shown in [7, Appendix A] by Demailly that the global log canonical threshold of a non-singular Fano variety equalled that of its *alpha invariant*, introduced by Tian in the paper [76]. A purely algebraic version of this theorem was then proved in the paper [49] by Odaka and Sano. This set of ideas is important in linking complex and birational aspects of a variety's geometry. In particular, by combining [76, Theorem 2.1] with the above equivalence tells us that

$$\mathrm{lct}(F) > \frac{M}{M+1}$$

implies the existence of a Kähler-Einstein metric on F. We are interested in the case where the canonical threshold is equal to one, as this then implies Birational superrigidity in a straightforward way. However, the converse does not hold. In particular, it was proved that every non-singular index two hypersurface is *K-stable* by the paper [1], which implies the admission of a Kähler-Einstein metric. However, Birational superrigidity is a property that can only be held by index 1 varieties.

4.1.2. Relationship to Birational Rigidity — Note that in general, this is a much more powerful criteria than that used in Birational rigidity. In Birational rigidity, we only care about linear systems that are mobile. Indeed, we can similarly define the so-called *mobile canonical threshold* $\mathrm{mct}(X)$ to be:

$$\mathrm{mct}(X) = \sup\{\lambda \in \mathbb{Q}_+ | (X, \frac{\lambda}{n} D) \text{ is canonical for a general } D \in \Sigma \subset |-nH|\}$$

where Σ is an arbitrary mobile linear system. Clearly we have that $\mathrm{mct}(X) > \mathrm{ct}(X)$, so calculating the canonical threshold of X gives us a lower bound of the mobile canonical threshold. The connection to Birational Rigidity was first used for the following:

Theorem 4.1.3. — *[54, Theorem 1] Let $F_1, \ldots, F_K$, $K \geq 2$ be primitive Fano varieties, and suppose that the conditions $\mathrm{lct}(F_i) = 1$ and $\mathrm{mct}(F_i) \geq 1$ hold. Then their direct product*

$$V = F_1 \times \ldots \times F_K$$

is a birationally superrigid variety. In particular,

1. *Every structure of a rationally connected fibre space on the variety V is given by a projection onto a direct factor. More precisely, if $\beta : V^\sharp \to S^\sharp$ is a rationally connected fibre space and $\chi : V \dashrightarrow V^\sharp$ is an arbitrary birational map, then there exists a subset of indices*

$$I = \{i_i, \ldots, i_k\} \subset \{1, \ldots, K\}$$

and a birational map

$$\alpha : F_I = \prod_{i \in I} F_i \dashrightarrow S^\sharp$$

such that the diagram

$$\begin{tikzcd} V \arrow[r, dashed, "\chi"] \arrow[d, "\pi_I"'] & V^\sharp \arrow[d, "\beta"] \\ F_I \arrow[r, dashed, "\alpha"] & S^\sharp \end{tikzcd}$$

commutes, that is $\beta \circ \chi = \alpha \circ \pi_I$*, where* π_I *is the natural projection onto a direct factor:*

$$\pi_I : \prod_{i=1}^{K} F_i \to \prod_{i \in I} F_i.$$

2. *Let* $V^\sharp$ *be a variety with* $\mathbb{Q}$*-factorial terminal singularities satisfying the condition*

$$\dim_{\mathbb{Q}}(\operatorname{Pic} V^\sharp \otimes \mathbb{Q}) \leq K$$

and suppose $\chi : V \dashrightarrow V^\sharp$ *is a birational map. Then* χ *is a (biregular) isomorphism.*

3. *The groups of birational and biregular self-maps of the variety* V *coincide:*

$$\operatorname{Bir} V = \operatorname{Aut} V.$$

In particular, the group $\operatorname{Bir} V$ *is finite.*

4. *The variety* V *admits no structures of a fibration into rationally connected varieties of dimension strictly smaller than* $\min\{\dim F_i\}$*. In particular,* V *doesn't admit a structure of a conic bundle or a fibration into rational surfaces.*

5. *The variety* V *is non-rational.*

Indeed, we will show that cyclic covers described below will satisfy the conditions of Theorem 4.1.3.

4.2. Cyclic Covers

When we talk about the canonicity of cyclic covers, we now prefer to simplify things and insist that our variety is non-singular. Let $M \geq 12$, and let $G = G_m \subset \mathbb{P}^{M+1} = \mathbb{P}$

be a non-singular hypersurface of degree m. We then consider a cyclic cover

$$\sigma : F \to G$$

branched over a hypersurface $W \cap G$. In particular, $W = W_{Kl} \subset \mathbb{P}$ is a hypersurface of degree Kl. Continuing this construction, we end up with a complete intersection in the weighted projective space

$$\mathbb{P}^* = \mathbb{P}(\underbrace{1, \ldots, 1}_{M+2}, l)$$

given by equations

$$f(x_0, \ldots, x_{M+1}) = 0, \quad u^k = g(x_0, \ldots, x_{M+1})$$

where f and g are homogeneous polynomials of degrees m and Kl respectively, where u is the l-weighted variable. We now require firstly that $K(l-3) \geq 9$, and that $m+(K-1)l = M+1$, so that F is a non-singular (and hence factorial) primitive Fano variety with Picard group generated by the pullback of a hyperplane section on the base which is denoted H.

Let

$$\mathcal{F} \subset H^0(\mathbb{P}^*, \mathcal{O}_{\mathbb{P}}(m)) \times H^0(\mathbb{P}^*, \mathcal{O}_{\mathbb{P}}(Kl))$$

be the parameter space defining non-singular irreducible reduced cyclic covers, with defining pairs of polynomials $(f, g) \in \mathcal{F}$ denoted by $\underline{f}$.

Theorem 4.2.1. — *There is a non-empty Zariski open subset $\mathcal{F}_{\text{reg}} \subset \mathcal{F}$ such that for every variety $V \in \mathcal{F}_{\text{reg}}$ and every divisor $D \sim nH$ the pair $(V, \frac{1}{n}D)$ is canonical.*

Remark 4.2.2. — Notice that in this case, we prove a stronger statement than when we are talking about the rigidity of a variety - we require that *every* divisor satisfies this condition, rather than only divisors sitting inside a mobile linear system. This stronger property is known as the *divisorial canonicity* of a variety.

4.3. Exclusion of Maximal Singularities

We fix a cyclic cover $F \subset \mathcal{F}_{\mathrm{reg}}$ and assume that $D \sim nH$ is an effective divisor on F such that the pair $(F, \frac{1}{n}D)$ is not canonical. We aim to derive a contradiction from this statement, proving the theorem. To do this, we apply the projection method, first used in the case of an abelian cover of projective space in the paper [58].

4.3.1. Projection — We outline a method that allows us to treat our variety originally embedded in weighted projective space as if it were in the usual non-weighted kind, so that we may use the full range of hypertangent divisors when taking intersections. This works in our setup as follows:

Let $o^* = (0 : \ldots : 0 : 1) = (0^{M+2} : 1) \in \mathbb{P}^*$ be the unique singular point of the weighted projective space $\mathbb{P}^*$. Clearly $o^* \notin F$. Consider the projection

$$\pi_{\mathbb{P}^*} : \mathbb{P}^* \backslash \{o^*\} \dashrightarrow \mathbb{P},$$

given locally as $\pi_{\mathbb{P}^*}((x_0 : \ldots : x_{M+1} : u)) = (x_0 : \ldots : x_{M+1})$.

We use the following lemma:

Lemma 4.3.2. — *Let $\gamma(x_0, \ldots, x_{M+1})$ be a weighted polynomial of degree l. Then the equation $u = \gamma(x^*)$ defines a hypersurface $R_\gamma \subset \mathbb{P}^*$ that does not contain the point $o^* = (0 : \ldots : 0 : 1)$. The projection $\pi_{\mathbb{P}^*}|_{R_\gamma}$ is an isomorphism of R_γ and $\mathbb{P}$.*

Proof. This is obvious. □

Therefore, considering the affine chart $\{x_0 = 0\} \subset \mathbb{P}^*$ with the natural affine coordinates $z_i = x_i/x_0$ and $y = u/x_0^l$ the projection $\pi_{\mathbb{P}}$ takes the form

$$\mathbb{A}^{M+2}_{z_1,\ldots,z_{M+1},y} \to \mathbb{A}^{M+1}_{z_1,\ldots,z_{M+1}},$$

$$(z_1, \ldots, z_{M+1}, y) \mapsto (z_1, \ldots, z_{M+1}),$$

where $\mathbb{A} = \mathbb{A}^{M+1}_{z^*}$ is the affine chart $\{x_0 \neq 0\}$ in $\mathbb{P}$. Clearly the affine hypersurface $R_\gamma \cap \{x_0 \neq 0\}$ is given by the equation $y = h(z_1, \ldots, z_M) = \gamma(1, z_1, \ldots, z_M)$.

The point of this discussion is to note that that the complete intersection $F_\gamma = F \cap R_\gamma$ identifies naturally with a codimension 2 complete intersection $\Gamma = Q_f \cap Q_g$ in $\mathbb{P}$, and its intersection with the affine chart $\mathbb{A}$ identifies with a codimension 2 complete intersection in the same affine space. Note that we will occasionally abuse this notation, and will refer to the complete intersection embedded in the affine chart $\mathbb{A}$ by the same symbols $Q_f \cap Q_g$.

The advantage of doing this is that we remove the obstruction on the degree of hypertangent divisors from the weighting of the ambient projective space, giving us more "room" to get our required contradiction.

4.3.3. Non-canonical divisors — In more detail, returning to our divisor $D \sim nH$, we note that we may assume that D is prime. If there is a non-canonical singularity the centre of which is of positive dimension, then the pair

$$\left(\Gamma, \frac{1}{n} D_\Gamma\right),$$

where $D_\Gamma = D|_\Gamma$, is again non-canonical. If the centres are points, then taking a general polynomial passing through one of them yields a pair $(\Gamma, \frac{1}{n} D_\Gamma)$ that is even non-log canonical, though this is not required in this situation.

In either case, we obtain a non-singular codimension 2 complete intersection $\Gamma \subset \mathbb{P}^{M+1}$ where the defining polynomials which by abuse of notation we again call f and g (with the same decomposition as before) define non-singular hypersurfaces of degrees m and Kl respectively, as well as an effective divisor $D_\Gamma \sim nH_\Gamma$, where H_Γ is the class of a hyperplane section generating the group $\operatorname{Pic} \Gamma$, such that the pair

$(\Gamma, \frac{1}{n}D_\Gamma)$ is non-canonical. We work with this pair, replacing our original one. Let

$$CS\left(\Gamma, \frac{1}{n}D_\Gamma\right)$$

be the union of centres of non-canonical singularities of the pair $(\Gamma, \frac{1}{n}D)$.

Just as in the case of a rigid cyclic cover, we need to formulate the regularity conditions required to use the hypertangent divisors:

They are as follows:

- (N1): For any linear form

  $$\lambda(z_*) \notin \langle g_1, f_1\rangle$$

  the sequence of homogeneous polynomials

  $$\{f_1|_{\lambda=0}, f_2|_{\lambda=0}, \ldots, f_m|_{\lambda=0}\}$$
  $$\{g_1|_{\lambda=0}, g_2|_{\lambda=0}, \ldots, g_{M-3}|_{\lambda=0}\}$$

  is regular in the ring $\mathcal{O}_{o,\mathbb{P}^N}$.

- (N2): For any linear form $\lambda \notin \langle g_1\rangle$, the set

  $$\overline{\Gamma \cap \{g_1 = g_2 = 0\} \cap \{\lambda = 0\}}$$

  is irreducible and reduced, where the g_i and f_i are considered as the defining polynomials for Γ, rather than the original variety F.

We will prove that the set where the polynomials fail these conditions is closed in the overall parameter space in section 4.4.

4.3.4. Inversion of Adjunction — Note that since the pair (Γ, D_Γ) is not canonical, then there exists a divisor E over Γ satisfying the inequality

$$\nu_E(D_\Gamma) > na(E, \Gamma).$$

Let $B \subset \Gamma$ be the centre of the exceptional divisor E. The inequality

$$\operatorname{mult}_B D_\Gamma > n$$

clearly holds, from which by 1.3.18 we deduce that $\dim B \leq 1$. Suppose first of all that B is not contained within the singular locus $\operatorname{Sing}\Gamma$.

Consider a non-singular point $o \in B$ of general position. Let $\sigma : \Gamma^+ \to \Gamma$ be the blow up with exceptional divisor $E^+ = \sigma^{-1}(o) \cong \mathbb{P}^M$. Then for some hyperplane $\Theta \subset E^+$ the inequality

$$\operatorname{mult}_o D + \operatorname{mult}_\Theta D^+ > 2n \tag{4.1}$$

holds, where D^+ is the strict transform of the divisor D_Γ on F^+ (This is Proposition 9 of the paper [54]).

4.3.5. The subvariety of high multiplicity — Now we consider a general hyperplane section Δ of the complete intersection Γ containing the point o and cutting out the hyperplane Θ on E^+ so that $\Delta^+ \cap E^+ = \Theta$.

Lemma 4.3.6. — *The restriction $D_\Delta = D|_\Delta = (D \circ \Delta)$ of the divisor D on Δ satisfies the inequality*

$$\operatorname{mult}_o D_\Delta > 2n. \tag{4.2}$$

Proof. By the intersection theory lemma 1.3.16, we have the following:

$$(D^+ \circ \Delta^+) = D^+_\Delta + Z,$$

where Z is an effective divisor on E^+. Looking at the multiplicities yields

$$\operatorname{mult}_o D_\Delta = \operatorname{mult}_o D + \deg Z,$$

since $\operatorname{mult}_o \Delta = 1$. However, Z contains B with multiplicity at least $\operatorname{mult}_B D^+$, from which the statement follows in combination with equation 4.1. $\square$

Note that by linearity we can assume that D_Δ is prime. Let

$$\Delta_f = \{f_1|_\Delta = 0\}.$$

By the condition (N1) we have the equality $\mathrm{mult}_o \Delta_f = 2$, it is irreducible due to the Lefschetz theorem, and so by 4.2 we can conclude that $\Delta_f \neq D_\Delta$. Since $\mathrm{mult}_o \Delta_f = 2$. We can consider the scheme-theoretic intersection $(D_\Delta \circ \Delta_f) = Y_2$, and take an irreducible component Y_2^* with maximal value of $\frac{\mathrm{mult}_o}{\deg}$ which is an effective codimension 2 cycle satisfying the following:

$$\frac{\mathrm{mult}_o}{\deg} Y_2^* > \frac{4}{m \cdot Kl}.$$

We now consider the following. Let:

$$g_{\leq i} = g_1 + \ldots + g_i$$

for $i = 1, \ldots, Kl - 1$ and consider the restricted second hypertangent system

$$\Lambda_{2,g}^\Delta = \left| s_0 g_{\leq 2}|_\Delta + s_1 g_1|_\Delta \right|$$

where $s_o \in \mathbb{C}$ and s_1 runs through the space of linear forms in the variables z_*. By the condition (N2) the base locus $\mathrm{Bs}(\Lambda_{2,g}^\Delta)$ is irreducible and reduced, and by the condition (N1) it is of codimension 2 on Δ. Let $D_2 \in \Lambda_{2,g}$ be a general divisor. By the above, it is clear that it does not contain Y_2^*, and so we obtain another effective cycle $Y_3 = (Y_2^* \circ D_2)$. Again taking a component with maximal value of $\frac{\mathrm{mult}_o}{\deg}$ yields a codimension 4 cycle Y_3^* such that

$$\frac{\mathrm{mult}_o}{\deg} Y_3^* > \frac{6}{m \cdot Kl}.$$

Finally, consider the divisor $\Delta_g = \{g_1|_\Delta = 0\}$

Lemma 4.3.7. — *The subvariety Y_3^* is not contained in the divisor Δ_g.*

Proof. The base set of the hypertangent system $\Lambda^{\Delta}_{2,g}$ is

$$S_\Delta = \{g_1|_\Delta = g_2|_\Delta = 0\}.$$

It is irreducible and reduced, and hence

$$\deg S_\Delta = 2 \deg \Delta.$$

By the condition (N1) we have the equality

$$\mathrm{mult}_o\, S_\Delta = 6,$$

so that $Y_3^* \not\subset S_\Delta$. Note that a particular polynomial $s_0 g_{\leq 2} + s_1 g_1$ vanishes on Y_3^*, where $s_0 \neq 0$ by generality of the divisor D_2. Suppose that $Y_3^* \subset \Delta_g$. Then immediately this would imply the vanishing (restricted to Y_3^*) of $g_1|_{Y_3^*}$ and $g_{\leq 2}|_{Y_3^*}$, implying that

$$g_2|_{Y_3^*} = 0,$$

since $g_{\leq 2} = g_1 + g_2$. But this implies that $Y_3^* \subset S_\Delta$, which is false. □

By this lemma, we have shown that the effective cycle

$$Y_4 = (Y_3^* \circ \Delta_g)$$

of codimension 4 on Δ is well-defined. We can further assume it to be irreducible, and that it satisfies the inequality

$$\frac{\mathrm{mult}_o}{\deg} Y_4 > \frac{12}{m \cdot Kl}.$$

4.3.8. Using the technique of hypertangent divisors — At this point, we are nearly finished, and can take intersections with hypertangent divisors in the following way to finish off the proof of exclusion of a maximal singularity lying over a non-singular point. We intersect Y_4 with hypertangent divisors first corresponding to the polynomial f, followed by the polynomial g in the following way: we take general hypertangent divisors

$$D_5 \in \Lambda^{\Delta}_{2,f}, \ldots, D_{m+2} \in \Lambda^{\Delta}_{m-1,f}$$
$$D_{m+3} \in \Lambda^{\Delta}_{4,g}, \ldots, D_{M-3} \in \Lambda^{\Delta}_{M-3,g}.$$

By the condition (N1), on successively intersecting Y_4 and taking irreducible components of maximal value of $\frac{\mathrm{mult}_o}{\deg}$, we obtain an irreducible curve $C \subset \Gamma$ satisfying the inequality

$$\frac{\mathrm{mult}_o}{\deg} C > \frac{12}{m \cdot Kl} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{m}{m-1} \cdot \frac{5}{4} \cdot \ldots \cdot \frac{K(l-1)-3}{K(l-1)-4} = \frac{6}{Kl} \cdot \frac{(K-1)l-3}{4} \geq 1$$

by the condition on the variables K and l. This is a contradiction, and have hence concludes the proof of the theorem.

4.4. Regularity Conditions

Finally we come to the proof that failure of the regularity conditions is confined to a closed subset of the parameter space. Note that showing that violation of each condition can be checked separately, and so we divide into the two cases (N1) and (N2).

4.4.1. (N1) — As usual, we have to show that violation of the regularity conditions imposes at least M independent conditions on the coefficients of the polynomials in question. The complete intersection Γ is non-singular, hence the tangent space to Γ is given by

$$T_o\Gamma = \{f_1 = g_1 = 0\}.$$

Let us relabel the polynomials of the sequence, excluding the linear terms, as $p_1, p_2, \ldots, p_{M-3}$.

We restate the regularity condition (N1) in the following way: for any hyperplane $S \subset T_o\Gamma$, the sequence

$$p_1|_S, p_2|_S, \ldots, p_{M-3}|_S$$

is regular at the origin o. Let us fix an isomorphism $T_o\Gamma \cong \mathbb{C}^{M-1}$, and set $\mathbb{T} = \mathbb{P}(T_o\Gamma) \cong \mathbb{P}^{M-2}$. Set $\delta(i) = \deg p_i$ and let $\mathcal{P}_{a,M-2}$ be the space of homogeneous polynomials on $\mathbb{T}$, and

$$\mathcal{P}_{\mathbb{T}} = \prod_{i=1}^{M-3} \mathcal{P}_{\delta(i),M-2}.$$

Supposing all the polynomials p_i vanish on a line $L \subset \mathbb{T}$, then clearly the regularity condition is violated: we take any hyperplane $\mathbb{S} \supset L$. For that reason, the case when the set $\{p_1 = \ldots = p_{M-3} = 0\}$ contains a line is considered separately.

The case of a line. — Let $\mathcal{B}^{\text{line}} \subset \mathcal{P}_{\mathbb{T}}$ be a closed subset of tuples $(p_1, \ldots, p_{M-3})$ such that for some lie $L \subset \mathbb{T}$ we have

$$p_1|_L \equiv \ldots \equiv p_{M-3}|_L \equiv 0.$$

Proposition 4.4.2. — *The following inequality holds:* $\operatorname{codim}(\mathcal{B}^{line} \subset \mathcal{P}_{\mathbb{T}}) \geq M$.

Proof. We begin with the following lemma:

Lemma 4.4.3. — *The following inequality holds:*

$$\operatorname{codim}(\mathcal{B}^{line} \subset \mathcal{P}_{\mathbb{T}}) = \sum_{i=1}^{M-3} (\delta(i) + 1) - 2(M-3).$$

Proof. The first component in the right hand side is the codimension of the set of polynomials vanishing on a fixed line $L \subset \mathbb{T}$. We then subtract off dimension of the Grassmannian of lines. □

By [67, Lemma 3.2], this is bounded below by

$$\frac{1}{2}\left(\frac{(M-2)^2}{2} + M - 2\right) - 2$$

which is clearly greater than $2(M-3) + M$ for $M \geq 13$.

□

Finishing the proof — To conclude, let us fix a hyperplane $\mathbb{S} \subset \mathbb{T}$ and its isomorphism $\mathbb{S} \cong \mathbb{P}^{M-3}$. Set

$$\mathcal{P} = \prod_{i=1}^{M-3} \mathcal{P}_{\delta(i),\, M-2}.$$

Since the hyperplane $\mathbb{S}$ varies in a $(M-2)$-dimensional family, it is sufficient to show that the codimension of the set of tuples $(p_1, \ldots, p_{M-3}) \in \mathcal{P}$ such that the closed set

$$\{p_1 = \ldots = p_{M-3} = 0\}$$

has a component of positive dimension, which isn't a line, is of codimension at least $M + (M-2) = 2M-2$ in $\mathcal{P}$. Let $\mathcal{B}_i \subset \mathcal{P}$ be the set of tuples such that the closed set

$$\{p_1 = \ldots = p_{i-1} = 0\} \subset \mathbb{P}^{M-3} \tag{4.3}$$

is of codimension $(i-1)$ in $\mathbb{P}^{M-3}$ but for some irreducible component of this set, B, say, we have $p_i|_B \equiv 0$, and moreover, if $i = M-3$, then B is a curve of degree at least 2. Theorem 4.2.1 is then implied by the following:

Proposition 4.4.4. — *The following inequality holds:*

$$\operatorname{codim}(\mathcal{B}_i \subset \mathcal{P}) \geq 2M-2.$$

Proof. By the usual method of estimating the codimension first seen in [60, Proposition 1], for $k = 1, 2$, we obtain the estimate

$$\operatorname{codim}(\mathcal{B}_i \subset \mathcal{P}) \geq \binom{M-i}{2}.$$

The minimum for this expression is obtained at $i - 2$, at which point we can check that

$$\binom{M-2}{2} - 2M + 2 \geq 0$$

for $M \geq 12$. Therefore, we may assume that $i \geq 3$, so that $\delta(i) \geq 3$. Now let $\mathcal{B}_{i,b\subset\mathcal{P}}$ be the set of tuples such that the closed set 4.3 is of codimension $(i-1)$, and moreover, there is an irreducible component B of this set such that

$$\operatorname{codim}(\langle B\rangle \subset \mathbb{P}^{M-3}) = b,$$

where $b \in \{0, 1, \ldots, i-1\}$, $b \neq M-4$ and $p_i|_B \equiv 0$. Since

$$\mathcal{B}_i = \bigcup_{b=0}^{i-1} \mathcal{B}_{i,b},$$

it is sufficient to show the inequality

$$\operatorname{codim}(\mathcal{B}_{i,b} \subset \mathcal{P}) \geq 2M-2$$

for $i \geq 3$, $b \in \{0, \ldots, i-1\}$, $b \neq M-4$. Applying the technique of good sequences and associated subvarieties, which we do not delve into in this thesis, gives the estimate

$$\operatorname{codim}(\mathcal{B}_{i,b} \subset \mathcal{P}) \geq (M-1)(2b+3) - 2b^2 - 6b - 5.$$

The right hand side of this inequality is attained either at $b = 0$, when we obtain $3M - 8 \geq 2M - 2$, or at $b = i - 1$ if $i \leq M - 4$, or $b = M - 5$ if $i = M - 3$. In either case, once again the expression is not smaller than $2M - 2$. $\square$

4.4.5. (N2) — We prove this using Proposition 1.3 of the paper [67], which states that the codimension where the regularity conditions fail is greater than or equal to $\frac{1}{2}(M^2 - 15M + 40)$, which is clearly positive for $M \geq 12$.

This finishes the proof of Theorem 4.2.1.

5.

Fibre Spaces and Further Questions

In the final chapter, we will use the results from Chapters 3 and 4 to derive some more results about the Birational geometry of cyclic covers. In particular, we concern ourselves with a pencil of cyclic covers, and more general types of fibre space.

5.1. Geometry of Fibre Spaces

Thus far we have only been concerned with the Birational (super)rigidity of varieties considered on their own. Or rather, as a fibre space over a single point. In fact, we can also consider the more general case of a fibre space over a base variety.

We recall the following definitions:

Definition 5.1.1. — Consider a standard rationally connected fibre space X/S. We get an obvious inclusion $\pi^* A^i_{\text{mob}} S \subset A^i_{\text{mob}} X$. Furthermore, $A^1 V = \mathbb{R}[K_X] \oplus \pi^* A^1 S$. We say a standard Fano fibre space $\pi : X \to S$ satisfies the *K-condition* if

$$A^1_{\text{mob}} X \subset R_+[-K_X] \oplus \pi^* A^1_+ S.$$

In other words, it is equivalent to say that for any mobile linear system $|-nK_X + \pi^* A|$

the class A is pseudo-effective.

Definition 5.1.2. — We say a standard fibre space $\pi : X \to \mathbb{P}^1$ satisfies the *K^2-condition* if

$$K_X^2 \notin \operatorname{Int} A_+^2 X,$$

where A_+^2 denotes the set of effective classes of codimension 2.

The following proposition links the two conditions together.

Proposition 5.1.3. — *If a fibre space $\pi : X \to \mathbb{P}^1$ satisfies the K^2-condition, it satisfies the K-condition as well.*

Proof. If we take the self intersection of the divisor $|-nK_V + lF|$ where F is the class of a fibre and the integers n and l are strictly positive, we obtain

$$(-nK_V + lF)^2 = n^2 K_V^2 + 2nl(-K_V \cdot F).$$

Since $-K_V \cdot F$ is clearly pseudo-effective, this immediately implies that l is positive, which implies the K-condition. □

This gives us a good proxy to study structures of a rationally connected fibre space due to the following proposition:

Proposition 5.1.4. — *[59, Chapter 4, Section 3] Assume that a rationally connected fibre space $\pi : X \to S$ satisfies the K-condition. Then we have the following:*

1. *For the threshold of canonical adjunction of a mobile linear system $\Sigma \subset |-nK_X + \pi^* A|$ we have the equality $c(\Sigma, X) = n$.*

2. *If the mobile linear Σ satisfies $c(\Sigma, X) = 0$, then Σ is a π-pullback of a mobile linear system Σ_S on the base S.*

3. *If the variety X is birationally superrigid, so we have equality of virtual and actual thresholds of canonical adjunction on X, then for any birational map*

$\chi : X \dashrightarrow X'$ to a rationally connected fibre space $\pi' : X' \to S'$, there is a rational dominant map $\psi : S \dashrightarrow S'$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, dashed, "\chi"] \arrow[d, "\pi"'] & X' \arrow[d, "\pi'"] \\ S \arrow[r, dashed, "\epsilon"] & S'. \end{tikzcd}$$

We also have the following slightly weaker definition; we need it for the following theorem.

Definition 5.1.5. — Let $\pi : X \to \mathbb{P}^1$ be a standard fibre space, so that $\operatorname{Pic} X = \mathbb{Z}K_X \oplus \mathbb{Z}_F$, where F is the class of a fibre of the projection π. Assume further that

$$A^2 X = \mathbb{Z}K_X^2 \oplus \mathbb{Z}H_F$$

holds, where $H_F = (-K_X \cdot F)$, where F is the class of a fibre of the projection π. We say that X satisfies the K^2-condition *of depth* $\epsilon \geq 0$ if

$$K_X^2 - \epsilon H_F \notin \operatorname{Int} A_+^2 X.$$

5.1.6. Recap on Pencils of Cyclic Covers — In the paper [55], in a series of propositions the following theorem on the Birational rigidity of pencils of Fano cyclic covers was proved. We summarise the statement of the theorem as follows.

Suppose $a_* = \{0 = a_0 \leq a_1 \leq \ldots \leq a_{M+1}\}$ is a non-decreasing sequence of non-negative integers, $\mathcal{E} = \bigoplus_{i=0}^{M+1} \mathcal{O}_{\mathbb{P}^1}(a_i)$ a locally free sheaf on $\mathbb{P}^1$, and let $X = \mathbb{P}(\mathcal{E})$ be the corresponding projective bundle. Calculating the Picard group and canonical divisor, we obtain the following:

$$\operatorname{Pic} X = \mathbb{Z}L_X \oplus \mathbb{Z}R, \quad K_X = -(M+2)L_X + (a_X - 2)R,$$

where L_X is the class of the tautological sheaf, R the class of a fibre of the morphism $\pi_X : X \to \mathbb{P}^1$ and $a_X = a_1 + \ldots + a_{M+1}$, $L_X^{M+2} = a_X$. For some a_Q, $a_W \in \mathbb{Z}_+$ let

$$Q \sim mL_X + a_Q R, \quad W_X \sim K(lL_X + a_W R)$$

be divisors on X, where $Q \subset X$ is a non-singular subvariety, and $W = W_X \cap Q$ be a non-singular divisor on Q. Let

$$\sigma : V \to Q$$

be the K-sheeted cyclic cover of the variety Q branched over the divisor W. The projection $\pi_X|_Q$ will be denoted by π_Q, the projection $\pi_Q \circ \sigma : V \to \mathbb{P}^1$ by π. The fibre $\pi_Q^{-1}(t)$, $t \in \mathbb{P}^1$ will be denoted by G_t (or simply G when it is clear), and the fibre $\pi^{-1}(t) \subset V$ by the symbol F_t or F. Set $L_Q = L_X|_Q$ and $L = \sigma^* L_Q$ respectively.

Again we have for the Picard group and the canonical divisor the following:

$$\operatorname{Pic} V = \mathbb{Z}L \oplus \mathbb{Z}F, \quad K_V = -L + (a_X + a_Q + (K-1)a_W - 2)F.$$

It is easy to check the formulae $(L^M \cdot F) = mK$, $L^{M+1} = K(ma_X + a_Q)$ hold. From here we obtain $(-K_V \cdot L^M) = K((1-m)a_Q - m(K-1)a_W + 2m)$ and

$$(K_V^2 \cdot L^{M-1}) = K(-ma_X + (1-2m)a_Q - 2m(K-1)a_W + 4m).$$

We write the parameters of the cover V in the form

$$((a_1, \ldots, a_{M+1}), (a_Q, a_W))$$

and moreover, among the numbers $a_1, \ldots, a_{M+1}$ we specify only non-zero values, if there are any, else we write (0). Using this, we can verify which values are permitted when checking the K^2-condition. Once done, the following is proved:

Proposition 5.1.7. — *1. The variety V satisfies the strong K^2-condition if one of the following takes place:*

- $a_W \geq 1$,
- $a_W = 0$, $a_Q \geq 3$,
- $a_W = 0$, $a_Q = 2$, $a_X \geq 1$,
- $a_W = 0$, $a_Q = 1$, $a_X \geq 3$,
- $a_W = a_Q = 0$, $a_X \geq 4$.

2. *If $a_W = 0$, $a_Q = 2$, $a_W = 0$ then the variety V satisfies the K^2-condition of depth $\frac{2}{m}$.*

3. *If $a_W = 0$, $a_Q = 1$, then the variety V satisfies the K^2-condition of depth $\frac{1}{m}$ for $a_X = 2$ and depth $(1 + \frac{1}{m})$ for $a_X = 1$.*

4. *If $a_W = a_Q = 0$, then the variety V satisfies the K^2-condition of depth 1 for $a_X = 3$ and depth 2 for $a_X = 2$.*

We now formulate the main result. Assume that the cyclic cover V is sufficiently general in the family constructed above.

Theorem 5.1.8. — 1. *The variety V is birationally superrigid, the projection $\pi : V \to \mathbb{P}^1$ is the only structure of a rationally connected fibre space on V, and the groups of birational and biregular automorphisms of the variety V coincide if the integral parameters of the variety either satisfy any of the six conditions of the first part of the previous theorem, or are of one of the following six types:*

$$((2), (0, 0)),\ ((2), (1, 0)),\ ((1, 1), (1, 0)),\ ((3), (0, 0)),\ ((1, 2)(0, 0)),\ ((1, 1, 1)(0, 0)).$$

2. *The variety V of the type $((1, 1)(0, 0))$ is birationally superrigid. However, the K-condition does not hold: the linear system $|-K_V - F|$ is mobile and determines a rational map $\phi : V \dashrightarrow \mathbb{P}^1$, the fibres of which are rationally connected. On the variety V there are precisely two structures of a rationally connected fibre space: the projection π and the map ϕ. There exists a unique (up to a fibrewise isomorphism) fibration into Fano cyclic covers $\pi^+ : V^+ \to \mathbb{P}^1$*

of the same type $((1,1)(0,0))$ *and a birational isomorphism* $\chi : V \dashrightarrow V^+$ *biregular in codimension one, such that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, dashed, "\chi"] \arrow[d, dashed, "\phi"'] & V^+ \arrow[d, "\pi^+"] \\ \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1. \end{tikzcd}$$

The correspondence $V \to V^+$ *is an involution, that is,* $(V^+)^+ = V$.

3. *The variety* V *of the type* $((0),(2,0))$ *is birationally superrigid. Again however, the* K*-condition does not hold: the linear system* $|-mK_V - F|$ *is mobile and determines a rational map, the fibres of which are rationally connected. The group of birational self-maps* $\operatorname{Bir} V$ *is strictly larger than the groups of birational automorphisms: it contains a non-trivial birational involution* $\tau \in \operatorname{Bir} V \setminus \operatorname{Aut} V$ *and moreover,* $\operatorname{Bir} V \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/K\mathbb{Z})$, *where* $\mathbb{Z}/2\mathbb{Z} = \{id, \tau\}$. *On* V *there are precisely two structures of a rationally connected fibre space: the projection* π *and the rational map* $\pi \circ \tau : V \dashrightarrow \mathbb{P}^1$, *and moreover* $|-mK_V - F| = \tau_* |F|$.

Now note that by Theorem 3.1.2 we can expand the singularities allowed in the fibres. In fact, it is clear that we can do this far more generally, over many different classes of varieties which are locally complete intersections by using the generalised $4n^2$-inequality. Whenever we take fibre spaces over a variety, it is inevitable that singularities are picked up. Whereas previously there was a lot of case-by-case checking involved in proving the superrigidity of the fibres for a given class of fibre spaces, and we had to limit ourselves to the case where the singularities were quadratic and high enough rank, now we are in a position to deal with singular varieties with points of much higher multiplicity than before.

There is also an application of the second theorem of this thesis to Mori fibre spaces, namely the following.

Theorem 5.1.9. — *[56, Theorem 1]*

Let $\pi : X \to S$ *be a Mori fibre space such that*

- *every fibre* $F_s = \pi^{-1}(s)$, $s \in S$ *is a factorial Fano variety with terminal singularities and Picard group* $\operatorname{Pic} F_s = \mathbb{Z}K_s$*;*

- *for every effective divisor* $D_s \in |nK_S|$ *the pair* $(F_s, \frac{1}{n}D_s)$ *is log canonical, and for every mobile linear system* $\Sigma_s \subset |nK_s|$ *and every general divisor* $D_s \in \Sigma_s$ *the pair* $(F_s, \frac{1}{n}D_s)$ *is canonical;*

- *for every mobile family* $\bar{\mathcal{C}}$ *of curves sweeping out the base* S*, and a curve* $C \in \bar{\mathcal{C}}$ *the class of the following cycle of dimension* $\dim F_s$ *for any positive* $N \geq 1$

$$-N(K_V \circ \pi^{-1}(\bar{C})) - F_s$$

is not effective, that is, not rationally equivalent to an effective cycle of dimension $\dim F$*.*

Then every birational map $\chi : V \dashrightarrow V'$ *onto the total space of a rationally connected fibre space* $\pi' : V'/S'$ *is fibrewise, that is to say that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, dashed, "\chi"] \arrow[d, "\pi"'] & V^+ \arrow[d, "\pi^+"] \\ S \arrow[r, dashed] & S'. \end{tikzcd}$$

Indeed we would be able to say that we would be able to apply this theorem more specifically to our case of a cyclic cover. The problem is that although it is certainly true that non-singular cyclic covers of the type discussed satisfy the conditions of the theorem, aside from the trivial fibre space $F \times S$, any such space would have singular fibres, the log canonicity of such has not yet been proved, since Theorem 4.2.1 is only concerned with smooth covers. It should be possible to gain results in this direction, similar in spirit to the paper [62].

5.2. Projective Covers

On the other hand, we can also relax the condition that our cover is cyclic, and instead consider more general covers. Once done, we should be able to say something about more general complete intersections in weighted projective space. Following the methods of this thesis, in principle we should be able to tackle this problem and get some bound on the dimension, number of defining polynomials and bounds on any singularities to gain some knowledge of the Birational geometry of such varieties.

A first step in this direction is the following, the proof of which was the inspiration for Chapter 4.

Theorem 5.2.1. — *[58, Theorem 0.1] Let $\pi : F \to \mathbb{P}^N$ be a d-sheeted cover of N-dimensional complex projective space $\mathbb{P}^N$ where $N \geq 10$, $d \geq 5$, and F is embedded in weighted projective space $\mathbb{P}(\underbrace{1, \ldots, 1}_{M+1}, l)$ and $N = (d-1)l$. Then a Zariski general such cover is birationally superrigid.*

Indeed following the projection method outlined in Chapter 4, we should be able to bound the canonical threshold of an arbitrary cover over a hypersurface, at the cost of imposing stronger regularity conditions. Combining the two discussions in the last section, we should be able to generalise to the case where we allow quadratic singularities (of high enough rank). With this done, we would be able to prove the Birational rigidity of pencils of such varieties.

5.3. Higher Index Varieties

We can also study higher index varieties using the methods developed in this thesis. Unfortunately, it is easy to see that we cannot directly use the definitions of Birational rigidity to describe higher index cases. This is because we have (infinitely many) Fano fibre spaces induced by linear projections; we can see this by use of the adjunction formula. We are able however to say something about the index 2 case, following the paper [64]:

Theorem 5.3.1. — *Let $X_M \subset \mathbb{P}^{M+1}$ be a generic degree M hypersurface in $M+1$-dimensional projective space where $M \geq 16$. Let $\chi : X \dashrightarrow Y$ be a surjective birational map onto a rationally connected fibre space $\lambda : Y \to S$ where $S \neq \{\mathrm{pt}\}$.*

Then $S = \mathbb{P}^1$ and for some isomorphism $\mathbb{P}^1 \to S$ and some subspace $P \subset \mathbb{P}^{M+1}$ of codimension 2 we have

$$\lambda \circ \chi = \beta \circ \pi_P$$

where π_P is the induced projection from the linear space P.

It should be possible to use the methods of this thesis combined with those of the paper above to consider the case of an index 2 cyclic cover without too much trouble. Indeed, as the total dimension increases, we should be able to say more about hypersurfaces with a higher index. This is summed up in the following problem, formulated in [19] by De Fernex:

Problem 5.3.2. — *Find a non-trivial function $g(N)$ such that for a class of hypersurfaces $X_d \subset \mathbb{P}^N$ with $g(N) \leq d \leq N$, the only Fano fibre spaces birational to X_d are those induced by linear projections $\mathbb{P}^N \dashrightarrow \mathbb{P}^k$ with $0 \leq k \leq N - d$.*

In principle, we should at least be able to answer what $g(2)$ may well be using methods similar to these, however the calculations involved quickly become intractable as soon as we go beyond this point. We would have to make use of a different set of ideas to attack this problem.

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